CBSE MATHEMATICS

SOLVED ANNUAL EXAMINATIONS PAPERS

[For Class XII]

Year 2008 (All India & Delhi Region)

By

OP GUPTA

[Electronics & Communications Engineering]

[INDIRA AWARD WINNER]

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ABOUT OP GUPTA…
IN HIS OWN WORDS

Having taught Mathematics since 2006, I have devoted myself to this subject. No matter how I feel, I feel better when I teach. I am always delighted when students tell me they hated Mathematics before and could never learn it, but taking Mathematics with me has made it understandable and even a little enjoyable. Every book, study material or practice material I have written tries to teach serious Mathematics in a way that allows the students to learn Mathematics without being afraid. I prefer writing the Mathematical concepts in my books in interactive format, and in a bit easy way which makes learning them enjoyable that too without stress. My various works on Mathematics can be found on **www.theOPGupta.com**.

One minute old at this writing, my hobbies (besides Mathematics) till this moment are painting, photography, novel reading, writing poems, articles and listening to music. Being a thinker, I have penned down many articles which has been beneficial for many of my students. For such articles, one can visit my blog: **www.theOPGupta.blogspot.com**.

I am qualified as an **Electronics & Communications engineer**.

I have been honoured with the prestigious **INDIRA AWARD** by the Govt. of Delhi.

To me, teaching Mathematics is always a great joy. I hope I can give you some of this joy!

READ, IF YOU WISH TO KNOW ABOUT ME…,

EXAMINATION PAPERS – 2008

MATHEMATICS CBSE (Delhi)

CLASS – XII

By OP Gupta [Elect.& Comm. Engg., Indira Award Winner; +91-9650 350 480]

Time allowed: 3 hours **Maximum marks: 100**

General Instructions:

1. *All questions are compulsory.*
2. *The question paper consists of 29 questions divided into three sections-A, B and C. Section A comprises of 10 questions of one mark each, Section B comprises of 12 questions of four marks each and Section C comprises of 7 questions of six marks each.*
3. *All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.*
4. *There is no overall choice. However, internal choice has been provided in 4 questions of four marks each and 2 questions of six marks each. You have to attempt only one of the alternatives in all such questions.*
5. *Use of calculators is not permitted.*

Set–I

SECTION–A

1. If $f(x) = x + 7$ and $g(x) = x - 7$, $x \in R$, find $(fog)(7)$
2. Evaluate : $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$
3. Find the value of x and y if : $2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix} + \begin{bmatrix}y & 0\\1 & 2\end{bmatrix} = \begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$
4. Evaluate: $\begin{vmatrix}a+ib & c+id\\-c+id & a-ib\end{vmatrix}$
5. Find the cofactor of a_{12} in the following: $\begin{vmatrix}2 & -3 & 5\\6 & 0 & 4\\1 & 5 & -7\end{vmatrix}$
6. Evaluate: $\int \frac{x^2}{1+x^3}\,dx$
7. Evaluate: $\int_0^1 \frac{dx}{1+x^2}$
8. Find a unit vector in the direction of $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

9. Find the angle between the vectors $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$

10. For what value of λ are the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ perpendicular to each other?

SECTION–B

11. (*i*) Is the binary operation defined on set N, given by $a * b = \dfrac{a+b}{2}$ for all $a, b \in N$, commutative?

(*ii*) Is the above binary operation associative?

12. Prove the following:

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$$

13. Let $A = \begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix}$.

Express A as sum of two matrices such that one is symmetric and the other is skew symmetric.

OR

If $A = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$, verify that $A^2 - 4A - 5I = 0$

14. For what value of k is the following function continuous at $x = 2$?

$$f(x) = \begin{cases} 2x + 1 & ; x < 2 \\ k & ; x = 2 \\ 3x - 1 & ; x > 2 \end{cases}$$

15. Differentiate the following with respect to x : $\tan^{-1}\left(\dfrac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right)$

16. Find the equation of tangent to the curve $x = \sin 3t$, $y = \cos 2t$ at $t = \dfrac{\pi}{4}$

17. Evaluate: $\displaystyle\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$

18. Solve the following differential equation:

$$(x^2 - y^2)\, dx + 2xy\, dy = 0$$

given that $y = 1$ when $x = 1$

OR

Solve the following differential equation:

$$\frac{dy}{dx} = \frac{x(2y - x)}{x(2y + x)}, \quad \text{if } y = 1 \text{ when } x = 1$$

19. Solve the following differential equation : $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

20. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$, find a vector $\vec{c}$ such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 3$.

OR

If $\vec{a} + \vec{b} + \vec{c} = 0$ and $|\vec{a}| = 3, |\vec{b}| = 5$ and $|\vec{c}| = 7$, show that the angle between $\vec{a}$ and $\vec{b}$ is 60°.

21. Find the shortest distance between the following lines :

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \text{ and } \frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$$

OR

Find the point on the line $\dfrac{x+2}{3} = \dfrac{y+1}{2} = \dfrac{z-3}{2}$ at a distance $3\sqrt{2}$ from the point (1, 2, 3).

22. A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability distribution of number of successes.

SECTION–C

23. Using properties of determinants, prove the following :

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$$

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix}$$

24. Show that the rectangle of maximum area that can be inscribed in a circle is a square.

OR

Show that the height of the cylinder of maximum volume that can be inscribed in a cone of height h is $\dfrac{1}{3}h$.

25. Using integration find the area of the region bounded by the parabola $y^2 = 4x$ and the circle $4x^2 + 4y^2 = 9$.

26. Evaluate: $\displaystyle\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\,dx$

27. Find the equation of the plane passing through the point (– 1, – 1, 2) and perpendicular to each of the following planes:

$2x + 3y - 3z = 2$ and $5x - 4y + z = 6$

OR

Find the equation of the plane passing through the points (3, 4, 1) and (0, 1, 0) and parallel to the line $\dfrac{x+3}{2} = \dfrac{y-3}{7} = \dfrac{z-2}{5}$

28. A factory owner purchases two types of machines, A and B for his factory. The requirements and the limitations for the machines are as follows :

Machine	Area occupied	Labour force	Daily output (in units)
A	1000 m^2	12 men	60
B	1200 m^2	8 men	40

He has maximum area of 9000 m^2 available, and 72 skilled labourers who can operate both the machines. How many machines of each type should he buy to maximise the daily output?

29. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter, a car and a truck are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver.

Set–II

Only those questions, not included in Set I, are given

20. Solve for x : $\tan^{-1}(2x) + \tan^{-1}(3x) = \dfrac{\pi}{4}$.

21. Evaluate: $\displaystyle\int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x}\, dx.$

22. If $y = \sqrt{x^2+1} - \log\left(\dfrac{1}{x} + \sqrt{1 + \dfrac{1}{x^2}}\right)$, find $\dfrac{dy}{dx}$.

23. Using properties of determinants, prove the following :

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3.$$

24. Evaluate: $\displaystyle\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx.$

25. Using integration, find the area of the region enclosed between the circles $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$.

Set–III

Only those questions, not included in Set I and Set II, are given.

20. Solve for x : $\tan^{-1}\left(\dfrac{x-1}{x-2}\right) + \tan^{-1}\left(\dfrac{x+1}{x+2}\right) = \dfrac{\pi}{4}$

21. If $y = \cot^{-1}\left[\dfrac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$, find $\dfrac{dy}{dx}$

22. Evaluate: $\displaystyle\int_0^1 \cot^{-1}[1 - x + x^2]\, dx$

23. Using properties of determinants, prove the following :

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

24. Using integration, find the area lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$.

25. Using properties of definite integrals, evaluate the following: $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$

SOLUTIONS

Set–I

SECTION–A

1. Given $f(x) = x + 7$ and $g(x) = x - 7,\ x \in R$

$\text{fog}(x) = f(g(x)) = g(x) + 7 = (x-7) + 7 = x$

$\Rightarrow$ $(\text{fog})(7) = 7$.

2. $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right] = \sin\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right] = \sin\frac{\pi}{2} = 1$

3. $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} 2 & 6 \\ 0 & 2x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} 2+y & 6 \\ 1 & 2x+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

Comparing both matrices

$2 + y = 5$ and $2x + 2 = 8$

$\Rightarrow$ $y = 3$ and $2x = 6$

$\Rightarrow$ $x = 3,\ y = 3$.

4. $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

$= (a+ib)(a-ib) - (c+id)(-c+id)$

$= [a^2 - i^2b^2] - [i^2d^2 - c^2]$

$= (a^2 + b^2) - (-d^2 - c^2)$

$= a^2 + b^2 + c^2 + d^2$

5. Minor of a_{12} is $M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = -42 - 4 = -46$

Cofactor $C_{12} = (-1)^{1+2} M_{12} = (-1)^3 (-46) = 46$

6. Let $I = \int \frac{x^2}{1+x^3} dx$

Putting $1 + x^3 = t$

$\Rightarrow \quad 3x^2 dx = dt$

or $\quad x^2 dx = \frac{dt}{3}$

$\therefore \qquad I = \frac{1}{3}\int \frac{dt}{t} = \frac{1}{3} \log|t| + C$

$= \frac{1}{3} \log|1 + x^3| + C$

7. $\int_0^1 \frac{dx}{1+x^2}$

$= \tan^{-1} x \Big|_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$

$= \frac{\pi}{4} - 0 = \frac{\pi}{4}.$

8. $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

Unit vector in direction of $\vec{a} = \frac{\vec{a}}{|\vec{a}|}$

$= \frac{3\hat{i} - 2\hat{j} + 6\hat{k}}{\sqrt{3^2 + (-2)^2 + 6^2}} = \frac{1}{7}(3\hat{i} - 2\hat{j} + 6\hat{k})$

9. $\vec{a} = \hat{i} - \hat{j} + \hat{k} \quad \Rightarrow \quad |\vec{a}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$

$\vec{b} = \hat{i} + \hat{j} - \hat{k} \quad \Rightarrow \quad |\vec{b}| = \sqrt{(1)^2 + (1)^2 + (-1)^2} = \sqrt{3}$

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$\Rightarrow \quad 1 - 1 - 1 = \sqrt{3}.\sqrt{3} \cos\theta \quad \Rightarrow \quad -1 = 3\cos\theta$

$\Rightarrow \quad \cos\theta = -\frac{1}{3} \quad \Rightarrow \quad \theta = \cos^{-1}\left(-\frac{1}{3}\right)$

10. $\vec{a}$ and $\vec{b}$ are perpendicular if

$\vec{a}.\vec{b} = 0$

$\Rightarrow \quad (2\hat{i} + \lambda\hat{j} + \hat{k}).(\hat{i} - 2\hat{j} + 3\hat{k}) = 0$

$\Rightarrow \quad 2 - 2\lambda + 3 = 0 \quad \Rightarrow \quad \lambda = \frac{5}{2}.$

SECTION–B

11. (*i*) Given N be the set

$$a * b = \frac{a+b}{2} \; \forall \; a, b \in N$$

To find * is commutative or not.

Now, $a * b = \dfrac{a+b}{2} = \dfrac{b+a}{2}$ $\quad \therefore$ (addition is commulative on N)

$= b * a$

So $\quad a * b = b * a$

$\therefore \quad$ * is commutative.

(*ii*) To find $a * (b * c) = (a * b) * c$ or not

$$\text{Now } a * (b * c) = a * \left(\frac{b+c}{2}\right) = \frac{a + \left(\frac{b+c}{2}\right)}{2} = \frac{2a+b+c}{4} \qquad ...(i)$$

$$(a * b) * c = \left(\frac{a+b}{2}\right) * c = \frac{\frac{a+b}{2} + c}{2}$$

$$= \frac{a+b+2c}{4} \qquad ...(ii)$$

From (*i*) and (*ii*)

$(a * b) * c \neq a * (b * c)$

Hence the operation is not associative.

12. L.H.S. $= \tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{1}{8}$

$$= \tan^{-1}\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\times\frac{1}{5}} + \tan^{-1}\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}$$

$$= \tan^{-1}\frac{8}{14} + \tan^{-1}\frac{15}{55}$$

$$= \tan^{-1}\frac{4}{7} + \tan^{-1}\frac{3}{11} = \tan^{-1}\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\times\frac{3}{11}}$$

$$= \tan^{-1}\frac{65}{77-12} = \tan^{-1}\frac{65}{65} = \tan^{-1} 1 = \frac{\pi}{4} \qquad = \text{R.H.S}$$

13. We know that any matrix can be expressed as the sum of symmetric and skew symmetric.

So, $A = \dfrac{1}{2}(A^T + A) + \dfrac{1}{2}(A - A^T)$

or $A = P + Q$ where P is symmetric matrix and Q skew symmetric matrix.

$$P = \frac{1}{2}(A + A^T) = \frac{1}{2}\left\{\begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 0 \\ 2 & 1 & 6 \\ 5 & 3 & 7 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 6 & 5 \\ 6 & 2 & 9 \\ 5 & 9 & 14 \end{bmatrix} = \begin{bmatrix} 3 & 3 & \frac{5}{2} \\ 3 & 1 & \frac{9}{2} \\ \frac{5}{2} & \frac{9}{2} & 7 \end{bmatrix}$$

$$Q = \frac{1}{2}(A - A^T)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & 3 \\ 0 & 6 & 7 \end{bmatrix} + \begin{bmatrix} -3 & -4 & 0 \\ -2 & -1 & -6 \\ -5 & -3 & -7 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 0 & -2 & 5 \\ 2 & 0 & -3 \\ -5 & 3 & 0 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 0 & -1 & \frac{5}{2} \\ 1 & 0 & -\frac{3}{2} \\ -\frac{5}{2} & \frac{3}{2} & 0 \end{bmatrix}$$

OR

$$A = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$$

$\therefore \quad A^2 = A \times A$

$$\therefore \quad = \begin{bmatrix} 1\times1+2\times2+2\times2 & 1\times2+2\times1+2\times2 & 1\times2+2\times2+2\times1 \\ 2\times1+1\times2+2\times2 & 2\times2+1\times1+2\times2 & 2\times2+1\times2+2\times1 \\ 2\times1+2\times2+1\times2 & 2\times2+2\times1+1\times2 & 2\times2+2\times2+1\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$4A = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} \text{ and } 5I = \begin{bmatrix} 5\times1 & 0 & 0 \\ 0 & 5\times1 & 0 \\ 0 & 0 & 5\times1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^2-4A-5I=\begin{bmatrix} 9-4-5 & 8-8 & 8-8 \\ 8-8 & 9-4-5 & 8-8 \\ 8-8 & 8-8 & 9-4-5 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

14. For continuity of the function at $x = 2$

$$\lim_{h\to 0} f(2-h) = f(2) = \lim_{h\to 0} f(2+h)$$

Now, $\quad f(2-h) = 2(2-h)+1 = 5-2h$

$\therefore \quad \lim_{h\to 0} f(2-h) = 5$

Also, $\quad f(2+h) = 3(2+h)-1 = 5+3h$

$$\lim_{h\to 0} f(2+h) = 5$$

So, for continuity $f(2) = 5$.

$\therefore \quad k = 5.$

15. Let $\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = y$

$$y = \tan^{-1}\left(\frac{1-\dfrac{\sqrt{1-x}}{\sqrt{1+x}}}{1+\dfrac{\sqrt{1-x}}{\sqrt{1+x}}}\right)$$

$$\Rightarrow \quad y = \tan^{-1}1 - \tan^{-1}\left(\frac{\sqrt{1-x}}{\sqrt{1+x}}\right)$$

$$\frac{dy}{dx} = 0 - \frac{1}{1+\left(\dfrac{\sqrt{1-x}}{\sqrt{1+x}}\right)^2}\cdot\frac{d}{dx}\left(\frac{\sqrt{1-x}}{\sqrt{1+x}}\right)$$

$$= -\frac{1+x}{2}\left\{\frac{\dfrac{-1}{2\sqrt{1-x}}\sqrt{1+x}-\dfrac{1}{2\sqrt{1+x}}\sqrt{1-x}}{1+x}\right\}$$

$$= \frac{1+x}{4}\left\{\frac{\dfrac{\sqrt{1+x}\times\sqrt{1+x}}{\sqrt{1-x}\times\sqrt{1+x}}+\dfrac{\sqrt{1-x}\times\sqrt{1-x}}{\sqrt{1+x}\times\sqrt{1-x}}}{1+x}\right\}$$

$$= \frac{1}{4}\cdot\frac{2}{\sqrt{1-x^2}} = \frac{1}{2\sqrt{1-x^2}}$$

16. Slope of tangent $= \dfrac{dy}{dx}$

$$= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d(\cos 2t)}{dt}}{\frac{d(\sin 3t)}{dt}} = \frac{-2\sin 2t}{3\cos 3t}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{\text{at } t = \frac{\pi}{4}} = \frac{-2\times\sin\frac{\pi}{2}}{3\times\cos\frac{3\pi}{4}} = \frac{-2\times 1}{3\times\left(-\frac{1}{\sqrt{2}}\right)} = \frac{2\sqrt{2}}{3}$$

Now $\quad x = \sin\left(\dfrac{3\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$

$$y = \cos\left(\frac{2\pi}{4}\right) = 0$$

$\therefore$ Equation of tangent is

$$y - 0 = \frac{dy}{dx}\left(x - \left(\frac{1}{\sqrt{2}}\right)\right)$$

$$y = \frac{2\sqrt{2}}{3}\left(x - \frac{1}{\sqrt{2}}\right)$$

$$y = \frac{2\sqrt{2}}{3}x - \frac{2}{3}$$

or $\quad 3y = 2\sqrt{2}\,x - 2.$

17. Let $I = \displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

Apply the property $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin x dx}{1+\cos^2 x}$$

$$I = \pi\int_0^{\pi} \frac{dx}{1+\cos^2 x} - I \qquad \Rightarrow \qquad 2I = \pi\int_0^{\pi} \frac{dx}{1+\cos^2 x}$$

$$I = \pi\int_0^{\pi/2} \frac{\sec^2 x}{1+\sec^2 x}\,dx \qquad \left[\text{Using } \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx\right]$$

$$I = \pi\int_0^{\pi/2} \frac{\sec^2 x}{2+\tan^2 x}\,dx$$

Putting $\quad \tan x = t \qquad$ *if* $\; x = 0, \; t = 0$

$\sec^2 x dx = dt \qquad$ *if* $\; x = \dfrac{\pi}{2}, \; t = \infty$

$$I = \pi \int_0^{\infty} \frac{dt}{(\sqrt{2})^2 + t^2}$$

$$I = \pi \left| \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) \right|_0^{\infty}$$

$$I = \frac{\pi}{\sqrt{2}}\left(\frac{\pi}{2}\right)$$

$$I = \frac{\pi^2}{2\sqrt{2}}$$

18. $(x^2 - y^2)\, dx + 2xy\, dy = 0$

$$\frac{dy}{dx} = -\frac{(x^2 - y^2)}{2xy} \qquad ...(i)$$

It is homogeneous differential equation.

Putting $y = ux \Rightarrow u + \frac{xdu}{dx} = \frac{dy}{dx}$

From (i) $u + x\frac{du}{dx} = -x^2\frac{(1-u^2)}{2x^2u} = -\left(\frac{1-u^2}{2u}\right)$

$$\Rightarrow \frac{xdu}{dx} = -\left[\frac{1-u^2}{2u} + u\right]$$

$$\Rightarrow \frac{xdu}{dx} = -\left[\frac{1+u^2}{2u}\right]$$

$$\Rightarrow \frac{2u}{1+u^2}du = -\frac{dx}{x}$$

Integrating both sides, we get

$$\Rightarrow \int \frac{2udu}{1+u^2} = -\int \frac{dx}{x}$$

$$\Rightarrow \log|1+u^2| = -\log|x| + \log C$$

$$\Rightarrow \log\left|\frac{x^2+y^2}{x^2}\right||x| = \log C$$

$$\Rightarrow \frac{x^2+y^2}{x} = C$$

$$\Rightarrow x^2 + y^2 = Cx$$

Given that $y = 1$ when $x = 1$

$\Rightarrow$ $1 + 1 = C \Rightarrow C = 2.$

$\therefore$ Solution is $x^2 + y^2 = 2x$.

OR

$$\frac{dy}{dx} = \frac{x(2y-x)}{x(2y+x)} \qquad ...(i)$$

Let $y = ux$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$\Rightarrow u + x.\frac{du}{dx} = \left(\frac{2u-1}{2u+1}\right) \qquad [\text{from}(i)]$$

$$x\frac{du}{dx} = \frac{2u-1}{2u+1} - u$$

$$x\frac{du}{dx} = \frac{2u-1-2u^2-u}{2u+1}$$

$$\Rightarrow \int \frac{2u+1}{u-1-2u^2}\,du = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{2u+1}{2u^2-u+1}\,du = -\int \frac{dx}{x}$$

Let $2u+1 = A(4u-1) + B$; $A = \frac{1}{2}$, $B = \frac{3}{2}$

$$\Rightarrow \frac{1}{2}\int \frac{4u-1}{2u^2-u+1}\,du + \int \frac{\frac{3}{2}}{2u^2-u+1}\,du = -\log x + k$$

$$\Rightarrow \frac{1}{2}\log(2u^2-u+1) + \frac{3}{4}\int \frac{du}{\left(u-\frac{1}{4}\right)^2 + \frac{7}{16}} = -\log x + k$$

$$\log(2u^2-u+1) + \frac{3}{2}\,\frac{1}{\sqrt{7}/4}\tan^{-1}\left[\frac{\left(u-\frac{1}{4}\right)}{\frac{\sqrt{7}}{4}}\right] = -2\log x + k'$$

Putting $u = \frac{y}{x}$ and then $y = 1$ and $x = 1$, we get

$$k' = \log 2 + \frac{6}{\sqrt{7}}\tan^{-1}\frac{3}{\sqrt{7}}$$

$\therefore$ Solution is $\log\left(\frac{2y^2-xy+x^2}{x^2}\right) + \frac{6}{\sqrt{7}}\tan^{-1}\left(\frac{4y-x}{\sqrt{7}x}\right) + 2\log x = \log 2 + \frac{6}{\sqrt{7}}\tan^{-1}\frac{3}{\sqrt{7}}$

19. $\cos^2 x\frac{dy}{dx} + y = \tan x$

$$\frac{dy}{dx} + \sec^2 x \times y = \sec^2 x\tan x$$

It is a linear differential equation.

Integrating factor $= e^{\int \sec^2 x\, dx}$

$= e^{\tan x}$

General solution : $y.\, IF = \int Q.\, IF\, dx$

$$y.\, e^{\tan x} = \int e^{\tan x}\, .\, \tan x.\, \sec^2 x\, dx$$

Putting $\tan x = t \quad \Rightarrow \sec^2 x\, dx = dt$

$\therefore \quad ye^{\tan x} = \int e^t\, .\, t\, .\, dt$

$= e^t\, .\, t - \int e^t\, dt = e^t\, .\, t - e^t + k$

$= e^{\tan x}\, (\tan x - 1) + k$

$\therefore \quad y.\, e^{\tan x} = e^{\tan x}\, (\tan x - 1) + k$

where k is some constant.

20. Given $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j} - \hat{k}$

Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \hat{i}\,(z - y) + \hat{j}\,(x - z) + \hat{k}\,(y - x)$$

Given $\vec{a} \times \vec{c} = \vec{b}$

$(z - y)\hat{i} + (x - z)\hat{j} + (y - x)\,\hat{k} = \hat{j} - \hat{k}.$

Comparing both sides

$z - y = 0 \quad \therefore \quad z = y$

$x - z = 1 \quad \therefore \quad x = 1 + z$

$y - x = -1 \quad \therefore \quad y = x - 1$

Also, $\quad \vec{a}\, .\, \vec{c} = 3$

$(\hat{i} + \hat{j} + \hat{k})\, .\, (x\hat{i} + y\hat{j} + z\hat{k}) = 3$

$x + y + z = 3$

$(1 + z) + z + z = 3$

$3z = 2 \quad \therefore \quad z = 2\, /\, 3$

$y = 2\, /\, 3$

$x = 1 + \dfrac{2}{3} = \dfrac{5}{3}$

$\vec{c} = \dfrac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k})$

OR

$$\vec{a}+\vec{b}+\vec{c}=0$$

$\Rightarrow \quad (\vec{a}+\vec{b})^2=(-\vec{c})^2$

$\Rightarrow \quad (\vec{a}+\vec{b}).(\vec{a}+\vec{b})=\vec{c}.\vec{c}$

$\Rightarrow \quad |\vec{a}|^2+|\vec{b}|^2+2\vec{a}.\vec{b}=|\vec{c}|^2$

$\Rightarrow \quad 9+25+2\vec{a}.\vec{b}=49$

$\Rightarrow \quad 2\vec{a}.\vec{b}=49-25-9$

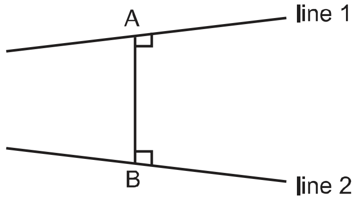
$\Rightarrow \quad 2|\vec{a}||\vec{b}|\cos\theta=15$

$\Rightarrow \quad 30\cos\theta=15$

$\Rightarrow \quad \cos\theta=\frac{1}{2}=\cos 60°$

$\Rightarrow \quad \theta=60°$

21. Let $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}=\lambda$ and $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}=k$



Now, let's take a point on first line as

$A(\lambda+3, -2\lambda+5, \lambda+7)$ and let

$B(7k-1, -6k-1, k-1)$ be point on the second line

The direction ratio of the line AB

$7k-\lambda-4, -6k+2\lambda-6, k-\lambda-8$

Now as AB is the shortest distance between line 1 and line 2 so,

$(7k-\lambda-4)\times 1+(-6k+2\lambda-6)\times(-2)+(k-\lambda-8)\times 1=0$ ...(*i*)

and $(7k-\lambda-4)\times 7+(-6k+2\lambda-6)\times(-6)+(k-\lambda-8)\times 1=0$ ...(*ii*)

Solving equation (*i*) and (*ii*) we get

$\lambda=0$ and $k=0$

$\therefore \quad A\equiv(3,5,7)$ and $B\equiv(-1,-1,-1)$

$\therefore \quad AB=\sqrt{(3+1)^2+(5+1)^2+(7+1)^2}=\sqrt{16+36+64}=\sqrt{116}$ units $=2\sqrt{29}$ units

OR

Let $\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{2}=\lambda$

$\therefore \quad (3\lambda-2, 2\lambda-1, 2\lambda+3)$ is any general point on the line

Now if the distance of the point from (1, 2, 3) is $3\sqrt{2}$, then

$\sqrt{(3\lambda-2-1)^2+(2\lambda-1-2)^2+(2\lambda+3-3)^2}=(3\sqrt{2})$

$\Rightarrow \quad (3\lambda-3)^2+(2\lambda-3)^2+4\lambda^2=18$

$\Rightarrow \quad 9\lambda^2-18\lambda+9+4\lambda^2-12\lambda+9+4\lambda^2=18$

$\Rightarrow \quad 17\lambda^2 - 30\lambda = 0$

$\Rightarrow \quad \lambda(17\lambda - 30) = 0$

$\Rightarrow \quad \lambda = 0 \quad \text{or} \quad \lambda = \frac{30}{17}$

$\therefore$ Required point on the line is $(-2,-1,3)\, or \left(\frac{56}{17}, \frac{43}{17}, \frac{77}{17}\right)$

22. Let X be the numbers of doublets. Then, $X = 0, 1, 2, 3$ or 4

$P(X = 0) = P$ (non doublet in each case)

$$P(\overline{D}_1\overline{D}_2\overline{D}_3\overline{D}_4) = \left(\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\right) = \frac{625}{1296}$$

$P(X = 1) = P$ (one doublet) $\left[\text{Alternatively use } {}^nC_r\, p^r\, q^r \text{ where } p = \frac{1}{6}, q = \frac{5}{6}\right]$

$$= P(D_1\overline{D}_2\overline{D}_3\overline{D}_4) \text{ or } P(\overline{D}_1D_2\overline{D}_3\overline{D}_4) \text{ or } P(\overline{D}_1\overline{D}_2D_3\overline{D}_4) \text{ or } P(\overline{D}_1\overline{D}_2\overline{D}_3D_4)$$

$$= \left(\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\right) + \left(\frac{5}{6}\times\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}\right) + \left(\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}\times\frac{5}{6}\right) + \left(\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}\right)$$

$$= \left(4\times\frac{125}{1296}\right) = \frac{125}{324}$$

$P(X = 2) = P$ (two doublets)

$$= P(D_1D_2\overline{D}_3\overline{D}_4) \text{ or } P(D_1\overline{D}_2D_3\overline{D}_4) \text{ or } P(D_1\overline{D}_2\overline{D}_3D_4) \text{ or } P(\overline{D}_1D_2D_3\overline{D}_4)$$
$$\text{or } P(\overline{D}_1D_2\overline{D}_3D_4) \text{ or } P(\overline{D}_1\overline{D}_2D_3D_4)$$

$$= \left(\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}\right) + \left(\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}\times\frac{5}{6}\right) + \left(\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}\right)$$
$$+ \left(\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}\right) + \left(\frac{5}{6}\times\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}\right) + \left(\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}\right)$$

$$= \left(6\times\frac{25}{1296}\right) = \frac{25}{216}$$

$P(X = 3) = P$ (three doublets)

$$= P(D_1D_2D_3\overline{D}_4) \text{ or } P(D_1D_2\overline{D}_3D_4) \text{ or } P(D_1\overline{D}_2D_3D_4) \text{ or } P(\overline{D}_1D_2D_3D_4)$$

$$= \left(\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}\right) + \left(\frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}\right) + \left(\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}\right) + \left(\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\right)$$

$$= \left(4\times\frac{5}{1296}\right) = \frac{5}{324}$$

$P(X = 4) = P$ (four doublets) $= P(D_1D_2D_3D_4)$

$$= \left(\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\right) = \frac{1}{1296}$$

Thus, we have

$X = x_i$	0	1	2	3	4
P_i	$\frac{625}{1296}$	$\frac{125}{324}$	$\frac{25}{216}$	$\frac{5}{324}$	$\frac{1}{1296}$

SECTION–C

23. L.H.S. $= \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$

Applying $R_3 \to R_3 + R_1$ and taking common $(\alpha+\beta+\gamma)$ from R_3.

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta-\alpha & \gamma-\alpha \\ \alpha^2 & \beta^2-\alpha^2 & \gamma^2-\alpha^2 \\ 1 & 0 & 0 \end{vmatrix} \qquad (\text{Applying } C_2 \to C_2 - C_1, C_3 \to C_3 - C_1)$$

$= (\alpha+\beta+\gamma)[(\gamma^2-\alpha^2)(\beta-\alpha)-(\gamma-\alpha)(\beta^2-\alpha^2)]$ (Expanding along R_3)

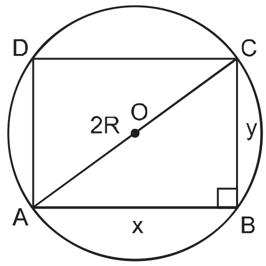
$= (\alpha+\beta+\gamma)(\gamma-\alpha)(\beta-\alpha)[(\gamma+\alpha)-(\beta+\alpha)]$

$= (\alpha+\beta+\gamma)(\gamma-\alpha)(\beta-\alpha)(\gamma-\beta)$

$= (\alpha+\beta+\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$

24. Let x and y be the length and breadth of rectangle and R be the radius of given circle, (*i.e.* R is constant).

Now, in right $\Delta\, ABC$, we have



$$x^2 + y^2 = (2R)^2$$

$$x^2 + y^2 = 4R^2 \Rightarrow y = \sqrt{4R^2 - x^2} \qquad ....(i)$$

Now, area, of rectangle $ABCD$.

$$A = xy$$

$$\Rightarrow \quad A = x\sqrt{4R^2 - x^2} \qquad [\text{from } (i)]$$

For area to be maximum or minimum

$$\frac{dA}{dx} = 0$$

$$\Rightarrow \quad x \times \frac{1}{2\sqrt{4R^2-x^2}} \times -2x + \sqrt{4R^2-x^2} \times 1 = 0$$

$$\Rightarrow \quad \frac{-x^2}{\sqrt{4R^2-x^2}} + \sqrt{4R^2-x^2} = 0 \quad \Rightarrow \quad \frac{(\sqrt{4R^2-x^2})^2 - x^2}{\sqrt{4R^2-x^2}} = 0$$

$$\Rightarrow \quad 4R^2 - x^2 - x^2 = 0 \quad \Rightarrow \quad 4R^2 - 2x^2 = 0$$

$$x^2 - 2R^2 = 0 \quad \Rightarrow \quad x = \sqrt{2}R$$

Now, $$\frac{d^2A}{dx^2} = \frac{2x(x^2 - 6R^2)}{(4R^2 - x^2)^{3/2}}$$

$\therefore$ $$\left.\frac{d^2A}{dx^2}\right|_{at\ x=\sqrt{2}R} = \frac{-8\sqrt{2}R^3}{(2R^2)^{3/2}} < 0$$

So, area will be maximum at $x = \sqrt{2}R$

Now, from (i), we have

$$y = \sqrt{4R^2 - x^2} = \sqrt{4R^2 - 2R^2} = \sqrt{2R^2}$$

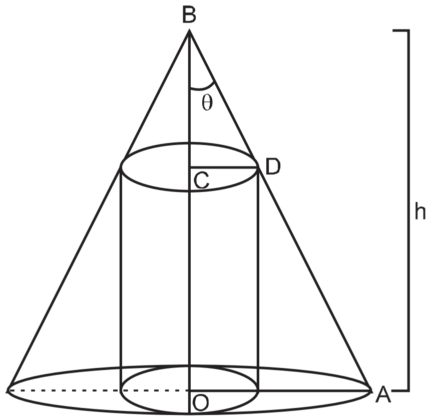
$$y = \sqrt{2}R$$

Here $x = y = \sqrt{2}R$

So the area will be maximum when $ABCD$ is a square.

OR

Let radius CD of inscribed cylinder be x and height OC be H and θ be the semi-vertical angle of cone.



Therefore,

$$OC = OB - BC$$

$$\Rightarrow \quad H = h - x\cot\theta$$

Now, volume of cylinder

$$V = \pi x^2 (h - x\cot\theta)$$

$$\Rightarrow \quad V = \pi(x^2 h - x^3 \cot\theta)$$

For maximum or minimum value

$$\frac{dV}{dx} = 0 \qquad \Rightarrow \qquad \pi(2xh - 3x^2\cot\theta) = 0$$

$$\Rightarrow \qquad \pi x(2h - 3x\cot\theta) = 0$$

$\therefore$ $$2h - 3x\cot\theta = 0 \qquad \text{(as } x = 0 \text{ is not possible)}$$

$$\Rightarrow \qquad x = \frac{2h}{3}\tan\theta$$

Now, $$\frac{d^2V}{dx^2} = \pi(2h - 6x\cot\theta)$$

$$\Rightarrow \qquad \frac{d^2V}{dx^2} = 2\pi h - 6\pi x\cot\theta$$

$$\Rightarrow \qquad \left.\frac{d^2V}{dx^2}\right|_{at\ x=\frac{2h\tan\theta}{3}} = 2\pi h - 6\pi \times \frac{2h}{3}\tan\theta\cot\theta$$

$$= 2\pi h - 4\pi h = -2\pi h < 0$$

Hence, volume will be maximum when $x = \frac{2h}{3} \tan \theta$.

Therefore, height of cylinder

$$H = h - x \cot \theta$$

$$= h - \frac{2h}{3} \tan \theta \cot \theta = h - \frac{2h}{3} = \frac{h}{3}.$$

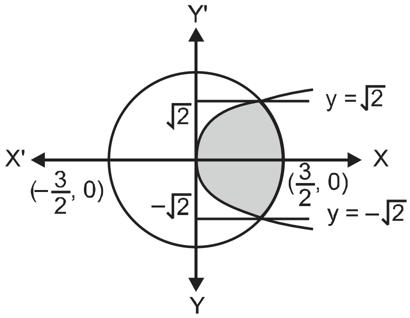
Thus height of the cylinder is $\frac{1}{3}$ of height of cone.

25. $x^2 + y^2 = \frac{9}{4}$ ...(*i*)

$y^2 = 4x$ ...(*ii*)

From (*i*) and (*ii*)

$$\left(\frac{y^2}{4}\right)^2 + y^2 = \frac{9}{4}$$



Let $y^2 = t$

$$\frac{t^2}{16} + t = \frac{9}{4}$$

$$t^2 + 16t = 36$$

$$t^2 + 18t - 2t - 36 = 0$$

$$t(t + 18) - 2(t + 18) = 0$$

$$(t - 2)(t + 18) = 0$$

$$t = 2, -18$$

$$y^2 = 2$$

$$y = \pm \sqrt{2}$$

Required area $= \int_{-\sqrt{2}}^{\sqrt{2}} (x_2 - x_1)\, dy$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\sqrt{\frac{9}{4} - y^2} - \frac{y^2}{4}\right) dy$$

$$= 2\int_0^{\sqrt{2}} \sqrt{\left(\frac{3}{2}\right)^2 - y^2}\, dy - \frac{2}{4}\int_0^{\sqrt{2}} y^2\, dy$$

$$= 2\left[\frac{y}{2}\sqrt{\frac{9}{4} - y^2} + \frac{9}{8}\sin^{-1}\frac{y}{3/2}\right]_0^{\sqrt{2}} - \frac{1}{2}\left(\frac{y^3}{3}\right)_0^{\sqrt{2}}$$

$$= 2\left[\frac{\sqrt{2}}{2}\sqrt{\frac{9}{4} - 2} + \frac{9}{8}\sin^{-1}\frac{2\sqrt{2}}{3}\right] - \frac{1}{6}\, 2\sqrt{2}$$

$$= \frac{1}{\sqrt{2}} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{\sqrt{2}}{3}$$

$$= \frac{1}{3\sqrt{2}} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \text{sq. units}$$

26. Let $I = \int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}\,dx$

Put $x = a\cos 2\theta$

$dx = a(-\sin 2\theta)\, 2d\theta$

If $x = a$, then $\cos 2\theta = 1$

$2\theta = 0$

$\theta = 0$

$x = -a$, $\cos 2\theta = -1$

$2\theta = \pi$

$\theta = \frac{\pi}{2}$

$$\therefore \quad I = \int_{\pi/2}^{0}\sqrt{\frac{a - a\cos 2\theta}{a + a\cos 2\theta}}\, a(-\sin 2\theta)\, 2d\theta$$

$$= \int_{0}^{\pi/2}\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\, 2a\sin 2\theta\, d\theta$$

$$= 2a\int_{0}^{\pi/2} 2\sin^2\theta\, d\theta = 2a\int_{0}^{\pi/2}(1-\cos 2\theta)\, d\theta$$

$$= 2a\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = 2a\left[\left(\frac{\pi}{2} - \frac{\sin\pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right]$$

$$= 2a\left[\left(\frac{\pi}{2} - 0\right)\right] = \pi a$$

27. Equation of the plane passing through $(-1, -1, 2)$ is

$a(x+1) + b(y+1) + c(z-2) = 0$ ...(*i*)

(*i*) is perpendicular to $2x + 3y - 3z = 2$

$\therefore \quad 2a + 3b - 3c = 0$ ...(*ii*)

Also (*i*) is perpendicular to $5x - 4y + z = 6$

$\therefore \quad 5a - 4b + c = 0$ ...(*iii*)

From (*ii*) and (*iii*)

$$\frac{a}{3-12} = \frac{b}{-15-2} = \frac{c}{-8-15} = k$$

$$\Rightarrow \quad \frac{a}{-9} = \frac{b}{-17} = \frac{c}{-23} = k$$

$$\Rightarrow \quad a = -9k, \quad b = -17k, \quad c = -23k$$

Putting in equation (i)

$$-9k(x+1)-17k(y+1)-23k(z-2)=0$$

$\Rightarrow$ $$9(x+1)+17(y+1)+23(z-2)=0$$

$\Rightarrow$ $$9x+17y+23z+9+17-46=0$$

$\Rightarrow$ $$9x+17y+23z-20=0$$

$\Rightarrow$ $$9x+17y+23z=20.$$

Which is the required equation of the plane.

OR

Equation of the plane passing through $(3, 4, 1)$ is

$$a(x-3)+b(y-4)+c(z-1)=0 \qquad ...(i)$$

Since this plane passes through $(0, 1, 0)$ also

$\therefore$ $$a(0-3)+b(1-4)+c(0-1)=0$$

or $$-3a-3b-c=0$$

or $$3a+3b+c=0 \qquad ...(ii)$$

Since (i) is parallel to

$$\frac{x+3}{2}=\frac{y-3}{7}=\frac{z-2}{5}$$

$\therefore$ $$2a+7b+5c=0 \qquad ...(iii)$$

From (ii) and (iii)

$$\frac{a}{15-7}=\frac{b}{2-15}=\frac{c}{21-6}=k$$

$\Rightarrow$ $$a=8k,\ b=-13k,\ c=15k$$

Putting in (i), we have

$$8k(x-3)-13k(y-4)+15k(z-1)=0$$

$\Rightarrow$ $$8(x-3)-13(y-4)+15(z-1)=0$$

$\Rightarrow$ $$8x-13y+15z+13=0.$$

Which is the required equation of the plane.

28. Let the owner buys x machines of type A and y machines of type B.

Then

$$1000x+1200y \le 9000 \qquad ...(i)$$

$$12x+8y \le 72 \qquad ...(ii)$$

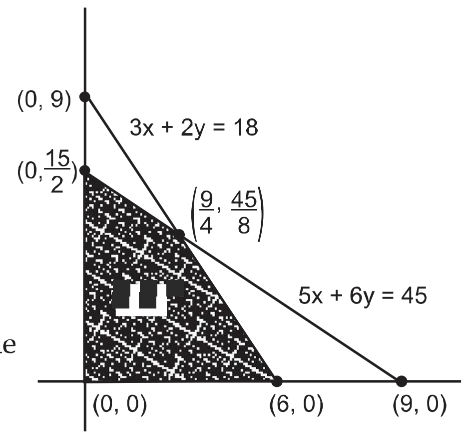
Objective function is to be maximize $z = 60x + 40y$

From (i)

$$10x+12y \le 90$$

or $$5x+6y \le 45 \qquad ...(iii)$$

$$3x+2y \le 18 \qquad ...(iv) \quad \text{[from } (ii)\text{]}$$



We plot the graph of inequations shaded region in the feasible solutions (iii) and (iv) .

The shaded region in the figure represents the feasible region which is bounded. Let us now evaluate Z at each corner point.

at (0, 0) Z is $60 \times 0 + 40 \times 0 = 0$

Z at $\left(0, \frac{15}{2}\right)$ is $60 \times 0 + 40 \times \frac{15}{2} = 300$

Z at $(6, 0)$ is $60 \times 6 + 40 \times 0 = 360$

Z at $\left(\frac{9}{4}, \frac{45}{8}\right)$ is $60 \times \frac{9}{4} + 40 \times \frac{45}{8} = 135 + 225 = 360.$

$\Rightarrow$ max. Z = 360

Therefore there must be

either $x = 6,\ y = 0$ or $x = \frac{9}{4}, y = \frac{45}{8}$ but second case is not possible as x and y are whole numbers. Hence there must be 6 machines of type A and no machine of type B is required for maximum daily output.

29. Let E_1 be the event that insured person is scooter driver,

E_2 be the event that insured person is car driver,

E_3 be the event that insured person is truck driver,

and A be the event that insured person meets with an accident.

$$\therefore \quad P(E_1) = \frac{2{,}000}{12{,}000} = \frac{1}{6}, \quad P\left(\frac{A}{E_1}\right) = 0.01$$

$$P(E_2) = \frac{4{,}000}{12000} = \frac{1}{3}, \quad P\left(\frac{A}{E_2}\right) = 0.03$$

$$P(E_3) = \frac{6{,}000}{12{,}000} = \frac{1}{2}, \quad P\left(\frac{A}{E_3}\right) = 0.15$$

$$\therefore \quad P\left(\frac{E_1}{A}\right) = \frac{P(E_1)\,.\,P\left(\frac{A}{E_1}\right)}{P(E_1)\,.\,P\left(\frac{A}{E_1}\right) + P(E_2)\,.\,P\left(\frac{A}{E_2}\right) + P(E_3)\,.\,P\left(\frac{A}{E_3}\right)}$$

$$= \frac{\frac{1}{6} \times 0.01}{\frac{1}{6} \times 0.01 + \frac{1}{3} \times 0.03 + \frac{1}{2} \times 0.15} = \frac{1}{1 + 6 + 45} = \frac{1}{52}$$

Set–II

20. We have,

$$\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{2x + 3x}{1 - (2x)\,.\,(3x)}\right] = \frac{\pi}{4} \quad \text{[Using property } \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}\text{]}$$

$\Rightarrow \quad \tan^{-1} \dfrac{5x}{1-6x^2} = \dfrac{\pi}{4}$

$\Rightarrow \quad \dfrac{5x}{1-6x^2} = 1 \quad \Rightarrow \quad 6x^2 + 5x - 1 = 0$

$\Rightarrow \quad 6x^2 + 6x - x - 1 = 0$

$\Rightarrow \quad 6x(x+1) - 1(x+1) = 0$

$\Rightarrow \quad (x+1)(6x-1) = 0$

$\Rightarrow \quad x = -1, \dfrac{1}{6}$ which is the required solution.

21. Let $I = \int_0^{\pi} \dfrac{x \tan x}{\sec x \operatorname{cosec} x}\, dx$

$\Rightarrow \quad I = \int_0^{\pi} \dfrac{x \cdot \dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos x} \cdot \dfrac{1}{\sin x}}\, dx$

$\Rightarrow \quad I = \int_0^{\pi} x \sin^2 x\, dx \qquad ...(i)$

$\Rightarrow \quad I = \int_0^{\pi} (\pi - x) \cdot \sin^2 (\pi - x)\, dx \qquad$ [Using property $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$]

$\Rightarrow \quad I = \int_0^{\pi} (\pi - x) \sin^2 x\, dx \qquad ...(ii)$

Adding (*i*) and (*ii*) we have

$2I = \int_0^{\pi} \pi \sin^2 x\, dx$

$\Rightarrow \quad 2I = \pi \int_0^{\pi} \sin^2 x\, dx = \dfrac{\pi}{2} \int_0^{\pi} (1 - \cos 2x)\, dx \quad \Rightarrow \quad 2I = \dfrac{\pi}{2}\left[x - \dfrac{\sin 2x}{2}\right]_0^{\pi}$

$\Rightarrow \quad 2I = \dfrac{\pi}{2}\left[\left(\pi - \dfrac{\sin 2\pi}{2}\right) - \left(0 - \dfrac{\sin 0}{2}\right)\right] \quad \Rightarrow \quad 2I = \dfrac{\pi}{2}[\pi] = \dfrac{\pi^2}{2}$

$\therefore \quad I = \dfrac{\pi^2}{4}.$

Hence $\int_0^{\pi} \dfrac{x \tan x}{\sec x \cdot \operatorname{cosec} x}\, dx = \dfrac{\pi^2}{4}.$

22. We have, $y = \sqrt{x^2+1} - \log\left(\dfrac{1}{x} + \sqrt{1 + \dfrac{1}{x^2}}\right)$

$\Rightarrow \quad y = \sqrt{x^2+1} - \log\left(\dfrac{1 + \sqrt{x^2+1}}{x}\right)$

$\Rightarrow \quad y = \sqrt{x^2+1} - \log\left(1 + \sqrt{x^2+1}\right) + \log x$

On differentiating w.r.t. x, we have

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+1}} \times 2x - \frac{1}{(\sqrt{x^2+1}+1)} \times \frac{1}{2\sqrt{x^2+1}} \times 2x + \frac{1}{x}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}\,(\sqrt{x^2+1}+1)} + \frac{1}{x}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}\,(\sqrt{x^2+1}+1)} \times \frac{(\sqrt{x^2+1}-1)}{(\sqrt{x^2+1}-1)} + \frac{1}{x}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x\left(\sqrt{x^2+1}-1\right)}{(\sqrt{x^2+1})\,(x^2)} + \frac{1}{x}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{(\sqrt{x^2+1}-1)}{x\sqrt{x^2+1}} + \frac{1}{x}$$

$$= \frac{x^2+1-\sqrt{x^2+1}+\sqrt{x^2+1}}{x\sqrt{x^2+1}}$$

$$= \frac{x^2+1}{x\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}}{x}$$

23. Let $\Delta = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

Applying $C_1 \to C_1 - b.C_3$ and $C_2 \to C_2 + a.C_3$, we have

$$\Delta = \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

Taking out $(1+a^2+b^2)$ from C_1 and C_2, we have

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

Expanding along first row, we have

$$= (1+a^2+b^2)^2\,[1.(1-a^2-b^2+2a^2) - 2b(-b)]$$

$$= (1+a^2+b^2)^2\,(1+a^2-b^2+2b^2)$$

$$= (1+a^2+b^2)^2\,(1+a^2+b^2) = (1+a^2+b^2)^3.$$

24. Let $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$ ...(*i*)

$\Rightarrow\ I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\, dx$ [Using property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$]

$\Rightarrow\ I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+(-\cos x)^2}\, dx$

$\Rightarrow\ I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\, dx$ ...(*ii*)

Adding (*i*) and (*ii*), we have

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\, dx = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$$

Let $\cos x = t \Rightarrow -\sin x\, dx = dt \Rightarrow \sin x\, dx = -dt$

As $x = 0, t = 1$ and $x = \pi, t = -1$

Now, we have

$$2I = \int_1^{-1} \frac{-dt}{1+t^2}$$

$$\Rightarrow\ 2I = \int_{-1}^{1} \frac{dt}{1+t^2} = [\tan^{-1}(t)]_{-1}^{1}$$

$$\Rightarrow\ 2I = \tan^{-1}(1) - \tan^{-1}(-1)$$

$$= \frac{\pi}{4} - \left(\frac{-\pi}{4}\right) = \frac{\pi}{2}$$

$$\Rightarrow\ I = \frac{\pi}{4}.$$

25. The equations of the given curves are

$x^2 + y^2 = 4$ ...(*i*)

and $(x-2)^2 + y^2 = 4$ ...(*ii*)

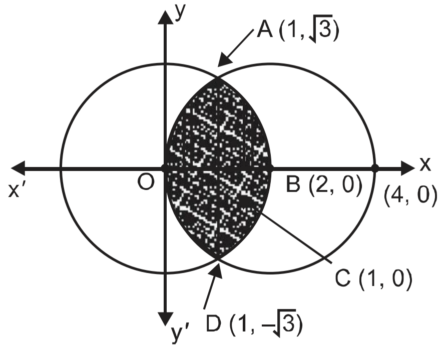
Clearly, $x^2 + y^2 = 4$ represents *a* circle with centre (0, 0) and radius 2. Also, $(x-2)^2 + y^2 = 4$ represents a circle with centre (2, 0) and radius 2. To find the point of intersection of the given curves, we solve (*i*) and (*ii*). Simultaneously, we find the two curves intersect at $A(1, \sqrt{3})$ and $D(1, -\sqrt{3})$.

Since both the curves are symmetrical about *x*-axis, So, the required area = 2(Area *OABCO*)

Now, we slice the area *OABCO* into vertical strips. We observe that the vertical strips change their character at $A(1, \sqrt{3})$. So,

Area *OABCO* = Area *OACO* + Area *CABC*.

When area $OACO$ is sliced in the vertical strips, we find that each strip has its upper end on the circle $(x-2)^2+(y-0)^2=4$ and the lower end on x-axis. So, the approximating rectangle shown in figure has length $= y_1$ width $= \Delta x$ and area $= y_1\ \Delta x$.



As it can move from $x = 0$ to $x = 1$

$\therefore$ Area $OACO = \int_0^1 y_1\ dx$

$\therefore$ Area $OACO = \int_0^1 \sqrt{4-(x-2)^2}\ dx$

Similarly, approximating rectangle in the region $CABC$ has length $= y_2$, width $= \Delta x$ and area $= y_2 \Delta x$.

As it can move from $x = 1$ to $x = 2$

$\therefore$ Area $CABC = \int_1^2 y_2 dx = \int_1^2 \sqrt{4-x^2}\ dx$

Hence, required area A is given by

$$A = 2\left[\int_0^1 \sqrt{4-(x-2)^2}\ dx + \int_1^2 \sqrt{4-x^2}\ dx\right]$$

$$\Rightarrow A = 2\left[\left[\frac{(x-2)}{2}.\sqrt{4-(x-2)^2}+\frac{4}{2}\sin^{-1}\frac{(x-2)}{2}\right]_0^1 + \left[\frac{x}{2}.\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_1^2\right]$$

$$\Rightarrow A = 2\left\{-\frac{\sqrt{3}}{2}+2\sin^{-1}\left(-\frac{1}{2}\right)-2\sin^{-1}(-1)+2\sin^{-1}(1)-\frac{\sqrt{3}}{2}-2\sin^{-1}\frac{1}{2}\right\}$$

$$= 2\left[-\sqrt{3}-2\left(\frac{\pi}{6}\right)+2\left(\frac{\pi}{2}\right)+2\left(\frac{\pi}{2}\right)-2\left(\frac{\pi}{6}\right)\right]$$

$$= 2\left(-\sqrt{3}-\frac{2\pi}{3}+2\pi\right)$$

$$= 2\left(\frac{4\pi}{3}-\sqrt{3}\right) = \left(\frac{8\pi}{3}-2\sqrt{3}\right) \text{ sq. units.}$$

Set–III

20. We have,

$$\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{\dfrac{x-1}{x-2}+\dfrac{x+1}{x+2}}{1-\left(\dfrac{x-1}{x-2}\right)\left(\dfrac{x+1}{x+2}\right)}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{(x-1)(x+2)+(x-2)(x+1)}{(x-2)(x+2)-(x-1)(x+1)}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right\}=\frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2x^2-4}{-3}\right)=\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{2x^2-4}{-3}=\tan\frac{\pi}{4} \qquad \Rightarrow \quad \frac{2x^2-4}{-3}=1$$

$$\Rightarrow \quad 2x^2-4=-3$$

$$\Rightarrow \quad 2x^2=1 \qquad \Rightarrow \quad x^2=\frac{1}{2} \qquad \Rightarrow \quad x=\pm\frac{1}{\sqrt{2}}$$

Hence, $x=\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}$ are the required values.

21. Given $y=\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$

$$=\cot^{-1}\left[\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}-\sqrt{1-\sin x})}\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}\right]$$

$$=\cot^{-1}\left[\frac{1+\sin x+1-\sin x+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}\right]$$

$$=\cot^{-1}\left[\frac{2(1+\cos x)}{2\sin x}\right]=\cot^{-1}\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}$$

$$=\frac{dy}{dx}=\frac{1}{2}$$

22. Let $I=\int_0^1 \cot^{-1}(1-x+x^2)\,dx$

$$=\int_0^1 \tan^{-1}\frac{1}{1-x+x^2}\,dx \qquad \left[\because \cot^{-1}x=\tan^{-1}\frac{1}{x}\right]$$

$$=\int_0^1 \tan^{-1}\frac{x+(1-x)}{1-x(1-x)}\,dx \qquad [\because 1 \text{ can be written as } x+1-x]$$

$$= \int_0^1 [\tan^{-1} x + \tan^{-1}(1-x)]\, dx \quad \left[\because \tan^{-1}\left\{\frac{a+b}{1-ab}\right\} = \tan^{-1} a + \tan^{-1} b\right]$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1-x)\, dx$$

$$= \int_0^1 \tan^{-1} x dx + \int_0^1 \tan^{-1}[1-(1-x)]\, dx \quad \left[\because \int_0^a f(x) = \int_0^a f(a-x)\, dx\right]$$

$$= 2\int_0^1 \tan^{-1} x dx = 2\int_0^1 \tan^{-1} x.1\, dx, \text{ integrating by parts, we get}$$

$$= 2\left[\{\tan^{-1} x.x\}_0^1 - \int_0^1 \frac{1}{1+x^2}\cdot x dx\right]$$

$$= 2[\tan^{-1} 1 - 0] - \int_0^1 \frac{2x}{1+x^2}\, dx = 2\cdot\frac{\pi}{4} - [\log(1+x^2)]_0^1$$

$$= \frac{\pi}{2} - (\log 2 - \log 1) = \frac{\pi}{2} - \log 2 \qquad [\because \log 1 = 0]$$

23. Let $\Delta = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we have

$$\Delta = \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

Taking out $2(a+b+c)$ from C_1, we have

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Interchanging row into column, we have

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a & b+c+2a & a \\ b & b & c+a+2b \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we have

$$\Delta = 2(a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ -(a+b+c) & a+b+c & a \\ 0 & -(a+b+c) & c+a+2b \end{vmatrix}$$

Now expanding along first row, we have

$$2(a+b+c)\,[1\,.\,(a+b+c)^2]$$

$$= 2(a+b+c)^3 \quad = \text{R.H.S.}$$

24. We have, given equations

$$x^2 + y^2 = 8x \qquad ...(i)$$

and $$y^2 = 4x \qquad ...(ii)$$

Equation (1) can be written as

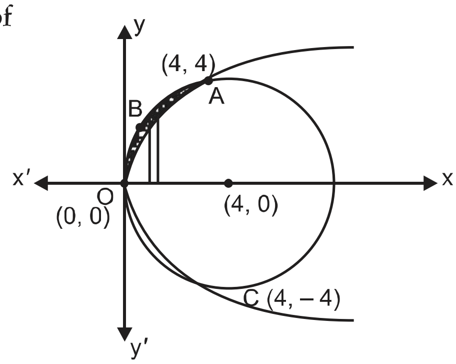
$$(x-4)^2 + y^2 = (4)^2$$

So equation (*i*) represents a circle with centre (4, 0) and radius 4.

Again, clearly equation (*ii*) represents parabola with vertex (0, 0) and axis as *x*-axis.

The curve (*i*) and (*ii*) are shown in figure and the required region is shaded.

On solving equation (*i*) and (*ii*) we have points of intersection 0(0, 0) and *A* (4, 4), *C*(4, – 4)



Now, we have to find the area of region bounded by (*i*) and (*ii*) & above *x*-axis.

So required region is *OBAO*.

Now, area of *OBAO* is

$$A = \int_0^4 \left(\sqrt{8x - x^2} - \sqrt{4x}\right) dx$$

$$= \int_0^4 \left(\sqrt{(4)^2 - (x-4)^2} - 2\sqrt{x}\right) dx$$

$$= \left[\frac{(x-4)}{2}\sqrt{(4)^2 - (x-4)^2} + \frac{16}{2}\sin^{-1}\frac{(x-4)}{4} - 2 \times \frac{2x^{3/2}}{3}\right]_0^4$$

$$= \left[8\sin^{-1} 0 - \frac{4}{3}(4)^{\frac{3}{2}}\right] - [8\sin^{-1}(-1) - 0]$$

$$= \left(8 \times 0 - \frac{4}{3} \times 8\right) - \left(8 \times -\frac{\pi}{2}\right)$$

$$= -\frac{32}{3} + 4\pi = \left(4\pi - \frac{32}{3}\right) \text{ sq.units}$$

25. Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\,dx \qquad ...(i)$

$$I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}\,dx \qquad \left[\text{Using property } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{-(\pi - x)\tan x}{-\sec x - \tan x}\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x}\, dx \qquad \text{...}(ii)$$

Adding (*i*) and (*ii*) we have

$$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \tan x}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\tan x}{(\sec x + \tan x)} \times \frac{(\sec x - \tan x)}{(\sec x - \tan x)}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\tan x(\sec x - \tan x)}{\sec^2 x - \tan^2 x}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} (\tan x \,.\, \sec x - \tan^2 x)\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} [\sec x \tan x - (\sec^2 x - 1)]\, dx$$

$$\Rightarrow \quad 2I = \pi\, [\sec x - \tan x + x]_0^{\pi}$$

$$\Rightarrow \quad 2I = \pi\, [(\sec \pi - \tan \pi + \pi) - (\sec 0 - \tan 0 + 0)]$$

$$\Rightarrow \quad 2I = \pi\, [(-1 - 0 + \pi) - (1 - 0)]$$

$$\Rightarrow \quad 2I = \pi\, (\pi - 2)$$

$$\therefore \quad I = \frac{\pi}{2}(\pi - 2)$$

Hence $\displaystyle\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} = \frac{\pi}{2}(\pi - 2)$

EXAMINATION PAPERS – 2008

MATHEMATICS CBSE (All India)

CLASS – XII

By OP Gupta [Elect.& Comm. Engg., Indira Award Winner; +91-9650 350 480]

Time allowed: 3 hours **Maximum marks: 100**

General Instructions: *As given in CBSE Examination paper (Delhi)* – 2008.

Set–I

SECTION–A

1. If $f(x)$ is an invertible function, find the inverse of $f(x) = \dfrac{3x-2}{5}$.

2. Solve for x : $\tan^{-1}\dfrac{1-x}{1+x} = \dfrac{1}{2}\tan^{-1}x;\ x > 0$

3. If $\begin{bmatrix} x+3y & y \\ 7-x & 4 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 4 \end{bmatrix}$, find the values of x and y.

4. Show that the points (1, 0), (6, 0), (0, 0) are collinear.

5. Evaluate : $\displaystyle\int \frac{x+\cos 6x}{3x^2+\sin 6x}\,dx$

6. If $\displaystyle\int (e^{ax}+bx)\,dx = 4e^{4x} + \frac{3x^2}{2}$, find the values of a and b.

7. If $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and angle between $\vec{a}$ and $\vec{b}$ is 60°, find $\vec{a}.\vec{b}$.

8. Find a vector in the direction of vector $\vec{a} = \hat{i} - 2\hat{j}$, whose magnitude is 7.

9. If the equation of a line AB is $\dfrac{x-3}{1} = \dfrac{y+2}{-2} = \dfrac{z-5}{4}$, find the direction ratios of a line parallel to AB.

10. If $\begin{vmatrix} x+2 & 3 \\ x+5 & 4 \end{vmatrix} = 3$, find the value of x.

SECTION–B

11. Let T be the set of all triangles in a plane with R as relation in T given by $R = \{(T_1, T_2) : T_1 \cong T_2\}$. Show that R is an equivalence relation.

12. Prove that $\tan\left(\dfrac{\pi}{4} + \dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right) + \tan\left(\dfrac{\pi}{4} - \dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right) = \dfrac{2b}{a}$.

OR

Solve $\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\dfrac{8}{31}$

13. Using properties of determinants, prove that following:

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

14. Discuss the continuity of the following function at $x = 0$:

$$f(x)=\begin{cases} \dfrac{x^4+2x^3+x^2}{\tan^{-1}x}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

OR

Verify Lagrange's mean value theorem for the following function:

$f(x)=x^2+2x+3,$ for $[4, 6]$.

15. If $f(x)=\sqrt{\dfrac{\sec x-1}{\sec +1}}$, find $f'(x)$. Also find $f'\left(\dfrac{\pi}{2}\right)$.

OR

If $x\sqrt{1+y}+y\sqrt{1+x}=0$, find $\dfrac{dy}{dx}$.

16. Show that $\int_0^{\pi/2} \sqrt{\tan x}+\sqrt{\cot x}=\sqrt{2}\pi$

17. Prove that the curves $x=y^2$ and $xy=k$ intersect at right angles if $8k^2=1$.

18. Solve the following differential equation:

$x\dfrac{dy}{dx}+y=x\log x;\ \ x\neq 0$

19. Form the differential equation representing the parabolas having vertex at the origin and axis along positive direction of x-axis.

OR

Solve the following differential equation:

$(3xy+y^2)dx+(x^2+xy)dy=0$

20. If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ are the position vectors of the points A, B, C and D, find the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Deduce that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

21. Find the equation of the line passing through the point $P(4, 6, 2)$ and the point of intersection of the line $\dfrac{x-1}{3}=\dfrac{y}{2}=\dfrac{z+1}{7}$ and the plane $x+y-z=8$.

22. A and B throw a pair of die turn by turn. The first to throw 9 is awarded a prize. If A starts the game, show that the probability of A getting the prize is $\dfrac{9}{17}$.

SECTION–C

23. Using matrices, solve the following system of linear equations:

$$2x - y + z = 3$$
$$-x + 2y - z = -4$$
$$x - y + 2z = 1$$

OR

Using elementary transformations, find the inverse of the following matrix:

$$\begin{bmatrix} 2 & -1 & 4 \\ 4 & 0 & 2 \\ 3 & -2 & 7 \end{bmatrix}$$

24. Find the maximum area of the isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with its vertex at one end of major axis.

OR

Show that the semi–vertical angle of the right circular cone of given total surface area and maximum volume is $\sin^{-1}\frac{1}{3}$.

25. Find the area of that part of the circle $x^2 + y^2 = 16$ which is exterior to the parabola $y^2 = 6x$.

26. Evaluate: $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$

27. Find the distance of the point $(-2, 3, -4)$ from the line $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$ measured parallel to the plane $4x + 12y - 3z + 1 = 0$.

28. An aeroplane can carry a maximum of 200 passengers. A profit of Rs. 400 is made on each first class ticket and a profit of Rs. 300 is made on each second class ticket. The airline reserves at least 20 seats for first class. However, at least four times as many passengers prefer to travel by second class then by first class. Determine how many tickets of each type must be sold to maximise profit for the airline. Form an LPP and solve it graphically.

29. A man is known to speak truth 3 out of 4 times. He throws a die and report that it is a 6. Find the probability that it is actually 6.

Set–II

Only those questions, not included in Set I, are given.

20. Using properties of determinants, prove the following:

$$\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9b^2(a+b)$$

21. Evaluate: $\int_0^{\pi/2} \log \sin x \, dx$

22. Solve the following differential equation:

$$(1 + x^2)\frac{dy}{dx} + y = \tan^{-1} x$$

27. Using matrices, solve the following system of linear equations:

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

OR

Using elementary transformations, find the inverse of the following matrix:

$$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 1 \\ 1 & 6 & 2 \end{bmatrix}$$

28. An insurance company insured 2000 scooter drivers, 3000 car drivers and 4000 truck drivers. The probabilities of their meeting with an accident respectively are 0.04, 0.06 and 0.15. One of the insured persons meets with an accident. Find the probability that he is a car driver.

29. Using integration, find the area bounded by the lines $x + 2y = 2$, $y - x = 1$ and $2x + y = 7$.

Set–III

Only those questions, not included in Set I and Set II are given.

20. If a, b and c are all positive and distinct, show that

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \text{ has a negative value.}$$

21. Evaluate: $\int_0^1 \cot^{-1}(1 - x + x^2) \, dx$

22. Solve the following differential equation:

$$x \log x \frac{dy}{dx} + y = 2 \log x$$

27. Using matrices, solve the following system of linear equations:

$$x + y + z = 4$$
$$2x + y - 3z = -9$$
$$2x - y + z = -1$$

OR

Using elementary transformations, find the inverse of the following matrix:

$$\begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 1 \\ 1 & 6 & 3 \end{bmatrix}$$

28. Find the area bounded by the curves $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$.

29. An insurance company insured 3000 scooter drivers, 5000 car drivers and 7000 truck drivers. The probabilities of their meeting with an accident respectively are 0.04, 0.05 and 0.15 One of the insured persons meets with an accident. Find the probability that he is a car driver.

SOLUTIONS

Set – I

SECTION–A

1. Given $f(x) = \dfrac{3x-2}{5}$

Let $\quad y = \dfrac{3x-2}{5}$

$\Rightarrow \quad 3x - 2 = 5y \quad \Rightarrow \quad x = \dfrac{5y+2}{3}$

$\Rightarrow \quad f^{-1}(x) = \dfrac{5x+2}{3}$

2. $\tan^{-1}\left(\dfrac{1-x}{1+x}\right) = \dfrac{1}{2}\tan^{-1} x$

$\Rightarrow \quad 2\tan^{-1}\left(\dfrac{1-x}{1+x}\right) = \tan^{-1} x$

$\Rightarrow \quad \tan^{-1}\dfrac{2\left(\dfrac{1-x}{1+x}\right)}{1-\left(\dfrac{1-x}{1+x}\right)^2} = \tan^{-1} x$

$\Rightarrow \quad \tan^{-1} 2\left(\dfrac{1-x}{1+x}\right)\dfrac{(1+x)^2}{(1+x)^2-(1-x)^2} = \tan^{-1} x$

$\Rightarrow \quad \tan^{-1}\dfrac{2(1+x)(1-x)}{4x} = \tan^{-1} x$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{1-x^2}{2x}\right) = \tan^{-1} x$

$\Rightarrow \quad \dfrac{1-x^2}{2x} = x \quad \Rightarrow \quad 1-x^2 = 2x^2$

$\Rightarrow \quad 3x^2 = 1 \quad \Rightarrow \quad x^2 = \frac{1}{3}$

$\Rightarrow \quad x = \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \quad \Rightarrow \quad x = \frac{1}{\sqrt{3}} \quad (\because x > 0)$

3. Given $\begin{bmatrix} x+3y & y \\ 7-x & 4 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 4 \end{bmatrix}$

Hence $x + 3y = 4$ ...(*i*)

$y = -1$ ...(*ii*)

$7 - x = 0$ ...(*iii*)

$\Rightarrow \quad x = 7, y = -1$

4. Since $\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$

Hence (1, 0), (6, 0) and (0, 0) are collinear.

5. Let $I = \int \frac{x + \cos 6x}{3x^2 + \sin 6x} dx$

Let $3x^2 + \sin 6x = t$

$\Rightarrow \quad (6x + 6\cos 6x)\, dx = dt$

$\Rightarrow \quad (x + \cos 6x)\, dx = \frac{dt}{6}$

$\therefore \quad I = \int \frac{dt}{6t} = \frac{1}{6} \log|t| + C = \frac{1}{6} \log|3x^2 + \sin 6x| + C$

6. $\int (e^{ax} + bx)\, dx = 4e^{4x} + \frac{3x^2}{2}$

Differentiating both sides, we get

$(e^{ax} + bx) = 16e^{4x} + 3x$

On comparing, we get $b = 3$

But a cannot be found out.

7. $|\vec{a}| = \sqrt{3}, \ |\vec{b}| = 2$

$\vec{a} . \vec{b} = |\vec{a}| . |\vec{b}| \cos\theta$

$= \sqrt{3} . 2 . \cos 60°$

$= \sqrt{3}$

8. $\vec{a} = \hat{i} - 2\hat{j}$

Unit vector in the direction of $\vec{a} = \frac{\hat{i} - 2\hat{j}}{\sqrt{5}}$

Hence a vector in the direction of $\vec{a}$ having magnitude 7 will be $\frac{7}{\sqrt{5}}\hat{i} - \frac{14}{\sqrt{5}}\hat{j}$.

9. The direction ratios of line parallel to AB is 1, –2 and 4.

10. $\begin{vmatrix} x+2 & 3 \\ x+5 & 4 \end{vmatrix} = 3$

$\Rightarrow \quad 4x + 8 - 3x - 15 = 3$

$\Rightarrow \quad x - 7 = 3$

$\Rightarrow \quad x = 10$

SECTION–B

11. (*i*) Reflexive

R is reflexive if $T_1 R T_1 \ \forall T_1$

Since $T_1 \cong T_1$

$\therefore$ R is reflexive.

(*ii*) Symmetric

R is symmetric if $T_1 R T_2 \Rightarrow T_2 R T_1$

Since $T_1 \cong T_2 \Rightarrow T_2 \cong T_1$

$\therefore$ R is symmetric.

(*iii*) Transitive

R is transitive if

$T_1 R T_2$ and $T_2 R T_3 \Rightarrow T_1 R T_3$

Since $T_1 \cong T_2$ and $T_2 \cong T_3 \Rightarrow T_1 \cong T_3$

$\therefore$ R is transitive

From (*i*), (*ii*) and (*iii*), we get

R is an equivalence relation.

12. L.H.S. $= \tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right)$

$$= \frac{\tan\frac{\pi}{4} + \tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1 - \tan\frac{\pi}{4}\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)} + \frac{\tan\frac{\pi}{4} - \tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1 + \tan\frac{\pi}{4}\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}$$

$$= \frac{1 + \tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1 - \tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)} + \frac{1 - \tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1 + \tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}$$

$$= \frac{\left[1 + \tan\left(\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right)\right]^2 + \left[1 - \tan\left(\frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)\right)\right]^2}{1 - \tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}$$

$$= \frac{2\sec^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1-\tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)} = \frac{2\sec^2\theta}{1-\tan^2\theta} = \frac{2(1+\tan^2\theta)}{1-\tan^2\theta} \qquad \left[\text{Let } \frac{1}{2}\cos^{-1}\left(\frac{a}{b}\right)=\theta\right]$$

$$= \frac{2}{\cos 2\theta} = \frac{2}{\cos 2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)} = \frac{2}{\frac{a}{b}}$$

$$= \frac{2b}{a} \quad = \text{R. H.S.}$$

OR

We have $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$

$\Rightarrow \quad \tan^{-1}\left[\frac{(x+1)+(x-1)}{1-(x^2-1)}\right] = \tan^{-1}\frac{8}{31}$

$\Rightarrow \quad \frac{2x}{2-x^2} = \frac{8}{31}$

$\Rightarrow \quad 62x = 16 - 8x^2$

$\Rightarrow \quad 8x^2 + 62x - 16 = 0$

$\Rightarrow \quad 4x^2 + 31x - 8 = 0$

$\Rightarrow \quad x = \frac{1}{4}$ and $x = -8$

As $x = -8$ does not satisfy the equation

Hence $x = \frac{1}{4}$ is only solution..

13. Let $\quad \Delta = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

Taking common $2(a+b+c)$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & 0 \\ 1 & 0 & c+a+2b \end{vmatrix} \qquad [\text{by } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1]$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

$= 2\,(a+b+\ c)\{(a+b+c)^2 - 0\}$ expanding along C_1.

$= 2(a+b+c)^3 = \text{RHS}$

14. At $x = 0$

L.H.L. $= \lim\limits_{h \to 0} \dfrac{(0-h)^4 + 2\,(0-h)^3 + (0-h)^2}{\tan^{-1}(0-h)}$

$= \lim\limits_{h \to 0} \dfrac{h^4 - 2h^3 + h^2}{-\tan^{-1} h} = \lim\limits_{h \to 0} \dfrac{h^3 - 2h^2 + h}{-\dfrac{\tan^{-1} h}{h}}$

[On dividing numerator and denominator by h.]

$= \dfrac{0}{-1}$ $\qquad \left(as \lim\limits_{h \to 0} \dfrac{\tan^{-1} h}{h} = 0\right)$

$= 0$

R.H.L $= \lim\limits_{h \to 0} \dfrac{(0+h)^4 + 2\,(0+h)^3 + (0+h)^2}{\tan^{-1}(0+h)}$

$= \lim\limits_{h \to 0} \dfrac{h^4 + 2h^3 + h^2}{\tan^{-1} h}$

$= \lim\limits_{h \to 0} \dfrac{h^3 + 2h^2 + h}{\dfrac{\tan^{-1} h}{h}}$ (on dividing numerator and denominator by h)

$= \dfrac{0}{1}$ $\qquad \left(as \lim\limits_{h \to 0} \dfrac{\tan^{-1} h}{h} = 1\right)$

$= 0$

and $f(0) = 0$ (given)

so, L.H.L = R.H.L = $f(0)$

Hence given function is continuous at $x = 0$

OR

$f(x) = x^2 + 2x + 3$ for $[4, 6]$

(i) Given function is a polynomial hence it is continuous

(ii) $f'(x) = 2x + 2$ which is differentiable

$f(4) = 16 + 8 + 3 = 27$

$f(6) = 36 + 12 + 3 = 51$

$\Rightarrow$ $f(4) \neq f(6)$. All conditions of Mean value theorem are satisfied.

$\therefore$ these exist atleast one real value $C \in (4,6)$

such that $f'(c) = \dfrac{f(6)-f(4)}{6-4} = \dfrac{24}{2} = 12$

$\Rightarrow$ $2c + 2 = 12$ or $c = 5 \in (4,6)$

Hence, Lagrange's mean value theorem is verified

15. $f(x) = \sqrt{\dfrac{\sec x - 1}{\sec x + 1}} = \sqrt{\dfrac{1-\cos x}{1+\cos x} \times \dfrac{1-\cos x}{1-\cos x}}$

$\Rightarrow$ $f(x) = \dfrac{1-\cos x}{\sin x} = \operatorname{cosec} x - \cot x$

$\Rightarrow$ $f'(x) = -\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x$

$\Rightarrow$ $f'(\pi/2) = -1 \times 0 + 1^2$

$\Rightarrow$ $f'(\pi/2) = 1$

OR

We have,

$x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow$ $x\sqrt{1+y} = -y\sqrt{1+x}$

$\Rightarrow$ $\dfrac{x}{y} = -\dfrac{\sqrt{x+1}}{\sqrt{1+y}}$

$\Rightarrow$ $\dfrac{x^2}{y^2} = \dfrac{x+1}{y+1}$

$\Rightarrow$ $x^2y + x^2 = xy^2 + y^2$

$\Rightarrow$ $x^2y - xy^2 + x^2 - y^2 = 0$

$\Rightarrow$ $xy(x-y) + (x-y)(x+y) = 0$

$\Rightarrow$ $(x-y)(xy + x + y) = 0$

but $x \neq y$ $\qquad \therefore$ $xy + x + y = 0$

$y(1+x) = -x$ $\qquad \therefore$ $y = \dfrac{-x}{1+x}$

$\therefore \dfrac{dy}{dx} = -\left[\dfrac{(1+x).1 - x \times 1}{(1+x)^2}\right]$ $\qquad = \dfrac{-1}{(1+x)^2}$

16. $\int_0^{\pi/2} \{\sqrt{\tan x} + \sqrt{\cot x}\} dx$

$\int_0^{\pi/2} \left(\dfrac{\sqrt{\sin x}}{\sqrt{\cos x}} + \dfrac{\sqrt{\cos x}}{\sqrt{\sin x}}\right) dx$

$$= \sqrt{2}\int_0^{\pi/2} \frac{(\sin x + \cos x)}{\sqrt{2\sin x \cos x}}\, dx = \sqrt{2}\int_0^{\pi/2} \frac{(\sin x + \cos x)}{\sqrt{1-(\sin x - \cos x)^2}}\, dx$$

Let $\sin x - \cos x = t$

$(\cos x + \sin x)\, dx = dt$

Now $x = 0 \Rightarrow t = -1$, and $x = \frac{\pi}{2} \Rightarrow t = 1$

$\therefore \quad \int_0^{\pi/2} \{\sqrt{\tan x} + \sqrt{\cot x}\}\, dx$

$$= \sqrt{2} \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = \sqrt{2} \left[\sin^{-1} t\right]_{-1}^{1}$$

$$= \sqrt{2}\,[\sin^{-1} 1 - \sin^{-1}(-1)]$$

$$= \sqrt{2}\,[2 \sin^{-1} 1].$$

$$= 2\sqrt{2}\left(\frac{\pi}{2}\right) = \sqrt{2}\,\pi = \text{RHS}$$

17. Given curves $\quad x = y^2 \quad$ ...(*i*)

$xy = k \quad$ ...(*ii*)

Solving (*i*) and (*ii*), $y^3 = k \;\therefore y = k^{1/3},\, x = k^{2/3}$

Differentiating (*i*) w. r. t. *x*, we get

$$1 = 2y \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(k^{2/3},\, k^{1/3})} = \frac{1}{2k^{1/3}} = m_1$$

And differentiating (*ii*) w.r.t. *x* we get

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(k^{2/3},\, k^{1/3})} = -\frac{k^{1/3}}{k^{2/3}} = -k^{-1/3} = m_2$$

$$\therefore \quad m_1 m_2 = -1$$

$$\Rightarrow \quad -\frac{1}{2k^{1/3}} \frac{1}{k^{1/3}} = -1 \quad \Rightarrow \quad k^{2/3} = 1/2 \quad \Rightarrow \quad 8k^2 = 1$$

18. Given $\quad x\frac{dy}{dx} + y = x \log x \quad$ ...(*i*)

$$\frac{dy}{dx} + \frac{y}{x} = \log x$$

This is linear differential equation

Integrating factor I.F. $= e^{\int \frac{1}{x} dx} = e^{\log_e x} = x$ Multiplying both sides of (*i*) by I.F. $= x$, we get

$$x\frac{dy}{dx} + y = x \log x$$

Integrating with respect to x, we get

$$y.\,x = \int x.\log x\, dx$$

$$\Rightarrow \quad xy = \log x.\frac{x^2}{2} - \int \frac{x^2}{2}.\frac{1}{x}\,dx$$

$$\Rightarrow \quad xy = \frac{x^2 \log x}{2} - \frac{1}{2}\frac{x^2}{2} + C$$

$$\Rightarrow \quad y = \frac{x}{2}\left(\log x - \frac{1}{2}\right) + C$$

19. Given $y^2 = 4ax$ ...(*i*)

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad y.\frac{dy}{dx} = 2a \qquad \therefore \qquad y\frac{dy}{dx} = 2.\frac{y^2}{4x} \qquad \text{(from } (i))$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{2x}$$ which is the required differential equation

OR

We have, $(3xy - y^2)dx + (x^2 + xy)\,dy = 0$

$$(3xy - y^2)dx = -(x^2 + xy)\,dy$$

$$\frac{dy}{dx} = \frac{y^2 - 3xy}{x^2 + xy}$$

Let $y = Vx$

$$\frac{dy}{dx} = \left(V + x\frac{dV}{dx}\right)$$

$$\therefore \quad \left(V + x\frac{dV}{dx}\right) = \frac{V^2x^2 - 3x.V.x}{x^2 + x.Vx}$$

$$\Rightarrow \quad V + x\frac{dV}{dx} = \frac{V^2 - 3V}{1+V}$$

$$\Rightarrow \quad x\frac{dV}{dx} = \frac{V^2 - 3V}{1+V} - V$$

$$\Rightarrow \quad x\frac{dV}{dx} = \frac{V^2 - 3V - V - V^2}{(1+V)} = \frac{-4V}{1+V}$$

$$\Rightarrow \quad \int \frac{1+V}{V}\,dV = -4\int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{1}{V}\,dV + \int dV = -4\int \frac{dx}{x}$$

$$\Rightarrow \quad \log V + V = -4\log x + C$$

$$\Rightarrow \quad \log V + \log x^4 + V = C$$

$$\Rightarrow \quad \log (V.x^4) + V = C$$

$$\Rightarrow \quad \log\left(\frac{y}{x}x^4\right) + \frac{y}{x} = C \quad \text{or } x\log(x^3y) + y = Cx$$

20. Given

$\overrightarrow{OA} = \hat{i} + \hat{j} + \hat{k}$

$\overrightarrow{OB} = 2\hat{i} + 5\hat{j}$

$\overrightarrow{OC} = 3\hat{i} + 2\hat{j} - 3\hat{k}$

$\overrightarrow{OD} = \hat{i} - 6\hat{j} - \hat{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \hat{i} + 4\hat{j} - \hat{k}$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = -2\hat{i} - 8\hat{j} + 2\hat{k}$

$\overrightarrow{CD} = -2(\hat{i} + 4\hat{j} - \hat{k})$

$\overrightarrow{CD} = -2\,\overrightarrow{AB}$

Therefore $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel vector so $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear and angle between them is zero.

21. Let $\frac{x-1}{3} = \frac{y}{2} = \frac{z+1}{7} = \lambda$ ...(*i*)

Coordinates of any general point on line (*i*) is of the form $\equiv (1 + 3\lambda, 2\lambda, -1 + 7\lambda)$

For point of intersection

$(1 + 3\lambda) + 2\lambda - (7\lambda - 1) = 8$

$1 + 3\lambda + 2\lambda - 7\lambda + 1 = 8$

$-2\lambda = 6$

$\lambda = -3$

Point of intersection $\equiv (-8, -6, -22)$

$\therefore$ Required equation of line passing through $P(4, 6, 2)$ and $Q(-8, -6, -22)$ is:

$$\frac{x-4}{4+8} = \frac{y-6}{6+6} = \frac{z-2}{2+22}$$

$$\therefore \quad \frac{x-4}{12} = \frac{y-6}{12} = \frac{z-2}{24}. \text{ or } x-4 = y-6 = \frac{z-2}{2}$$

22. Let E be the event that sum of number on two die is 9.

$E = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$

$$P(E) = \frac{4}{36} = \frac{1}{9}$$

$$P(E') = \frac{8}{9}$$

$$P \text{ (A getting the prize } P(A) = \frac{1}{9} + \frac{8}{9}\times\frac{8}{9}\times\frac{1}{9} + \frac{8}{9}\times\frac{8}{9}\times\frac{8}{9}\times\frac{8}{9}\times\frac{1}{9} + \ldots\ldots$$

$$= \frac{1}{9}\left(1 + \left(\frac{8}{9}\right)^2 + \left(\frac{8}{9}\right)^4 + \left(\frac{8}{9}\right)^6 + \ldots\right)$$

$$= \frac{1}{9}\frac{1}{\left[1-\left(\frac{8}{9}\right)^2\right]} = \frac{1}{9}\cdot\frac{9^2}{(9^2-8^2)} = \frac{9}{17}.$$

SECTION–C

23. Given System of linear equations

$$2x - y + z = 3$$
$$-x + 2y - z = -4$$
$$x - y + 2z = 1$$

we can write these equations as

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad AX = B, \text{ where, } A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B \qquad \ldots(i)$$

Now, $|A| = 2(4-1) - (-1)(-2+1) + 1(1-2)$

$= 6 - 1 - 1 = 4$

Again Co-factors of elements of matrix A are given by

$$C_{11} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - 1 = 3$$

$$C_{12} = -\begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} = -(-2+1) = 1$$

$$C_{13} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = (1 - 2) = -1$$

$$C_{21} = -\begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} = -(-2+1) = 1$$

$$C_{22} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = (4 - 1) = 3$$

$$C_{23} = -\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} = -(-2+1) = 1$$

$$C_{31} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = (1 - 2) = -1$$

$$C_{32} = -\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = -(-2+1) = 1$$

$$C_{33} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - 1 = 3$$

$$\therefore \quad \text{adj } A = (C)^T = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{adj.\, A}{|A|} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$\therefore$ From (i), we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ -8 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad x = 1, y = -2, z = -1$$

OR

$A = I_3 . A$

$$\begin{bmatrix} 2 & -1 & 4 \\ 4 & 0 & 2 \\ 3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 2 & -1 & 4 \\ 0 & 2 & -6 \\ 3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to 1/2R_1$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 2 \\ 0 & 2 & -6 \\ 3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 2 \\ 0 & 2 & -6 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -2 & 1 & 0 \\ \frac{-3}{2} & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 / 2$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 2 \\ 0 & 1 & -3 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{1}{2} & 0 \\ \frac{-3}{2} & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{1}{2} & 0 \\ -1 & \frac{1}{2} & 0 \\ \frac{-3}{2} & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 + 1/2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} -2 & \frac{1}{2} & 1 \\ -1 & \frac{1}{2} & 0 \\ -2 & \frac{1}{4} & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 6R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -2 & \frac{1}{2} & 1 \\ 11 & -1 & -6 \\ -2 & \frac{1}{4} & 1 \end{bmatrix} A$$

Applying $R_3 \to -2R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{1}{2} & 1 \\ 11 & -1 & -6 \\ 4 & -\frac{1}{2} & -2 \end{bmatrix} A$$

Hence $A^{-1} = \begin{bmatrix} -2 & \frac{1}{2} & 1 \\ 11 & -1 & -6 \\ 4 & -\frac{1}{2} & -2 \end{bmatrix}$

24. Let ΔABC be an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then coordinates of points A and B are given by $(a\cos\theta, b\sin\theta)$ and $(a\cos\theta, -b\sin\theta)$

The area of the isosceles $\Delta\, ABC = \frac{1}{2} \times AB \times CD$

$\Rightarrow \quad A(\theta) = \frac{1}{2} \times (2b\sin\theta) \times (a - a\cos\theta)$

$\Rightarrow \quad A(\theta) = ab\sin\theta(1 - \cos\theta)$

For A_{max}

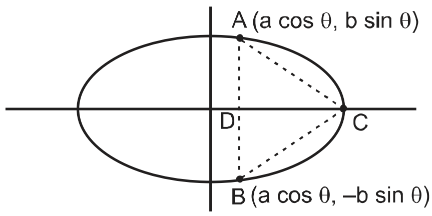
$$\frac{d(A(\theta))}{d\theta} = 0$$

$\Rightarrow \quad ab[\cos\theta(1 - \cos\theta) + \sin^2\theta] = 0$

$\cos\theta - \cos^2\theta + \sin^2\theta = 0$

$\Rightarrow \quad \cos\theta - \cos 2\theta = 0$

$\Rightarrow \quad \theta = \frac{2\pi}{3}$



Now, $\quad \frac{d^2(A(\theta))}{d\theta^2} = ab[-\sin\theta + 2\sin 2\theta]$

For $\quad \theta = \frac{2\pi}{3}, \frac{d^2(A(\theta))}{d\theta^2} = ab\left(-\frac{\sqrt{3}}{2} - 2 \times \frac{\sqrt{3}}{2}\right) < 0$

Hence for $\theta = \frac{2\pi}{3}$, A_{max} occurs

$\therefore \quad A_{max} = ab \sin \dfrac{2\pi}{3}\left(1 - \cos \dfrac{2\pi}{3}\right)$ square units

$= ab \dfrac{\sqrt{3}}{2}\left(1 + \dfrac{1}{2}\right) = \dfrac{3\sqrt{3}}{4} ab$ square units

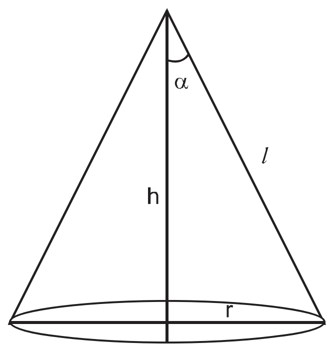
OR

Let r be the radius, l be the slant height and h be the vertical height of a cone of semi - vertical angle α.

Surface area $\quad S = \pi rl + \pi r^2 \quad$ ...(*i*)

or $\quad l = \dfrac{S - \pi r^2}{\pi r}$

The volume of the cone



$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2}$$

$$= \frac{\pi r^2}{3}\sqrt{\frac{(S - \pi r^2)^2}{\pi^2 r^2} - r^2}$$

$$= \frac{\pi r^2}{3}\sqrt{\frac{(S - \pi r^2)^2 - \pi^2 r^4}{\pi^2 r^2}}$$

$$= \frac{\pi r^2}{3} \times \frac{\sqrt{S^2 - 2\pi S r^2 + \pi^2 r^4 - \pi^2 r^4}}{\pi r} = \frac{r}{3}\sqrt{S(S - 2\pi r^2)}$$

$\therefore \quad V^2 = \dfrac{r^2}{9} S(S - 2\pi r^2) = \dfrac{S}{9}(Sr^2 - 2\pi r^4)$

$$\frac{dV^2}{dr} = \frac{S}{9}(2Sr - 8\pi r^3)$$

$$\frac{d^2V^2}{dr^2} = \frac{S}{9}(2S - 24\pi r^2) \quad \text{...(ii)}$$

Now $\quad \dfrac{dV^2}{dr} = 0$

$\Rightarrow \quad \dfrac{S}{9}(2Sr - 8\pi r^3) = 0 \quad$ or $\quad S - 4\pi r^2 = 0 \quad \Rightarrow \quad S = 4\pi r^2$

Putting $S = 4\pi r^2$ in (*ii*),

$$\frac{d^2V^2}{dr^2} = \frac{4\pi r^2}{9}[8\pi r^2 - 24\pi r^2] < 0$$

$\Rightarrow$ V is maximum when $S = 4\pi r^2$

Putting this value of S in (*i*)

$$4\pi r^2 = \pi rl + \pi r^2$$

or $\quad 3\pi r^2 = \pi rl$

or $\qquad \dfrac{r}{l} = \sin\alpha = \dfrac{1}{3}$

$\therefore \qquad \alpha = \sin^{-1}\left(\dfrac{1}{3}\right)$

Thus V is maximum, when semi vertical angle is $\sin^{-1}\left(\dfrac{1}{3}\right)$.

25. First finding intersection point by solving the equation of two curves

$$x^2 + y^2 = 16 \qquad \ldots(i)$$

and $\qquad y^2 = 6x \qquad \ldots(ii)$

$\Rightarrow \qquad x^2 + 6x = 16$

$\Rightarrow \qquad x^2 + 6x - 16 = 0$

$\Rightarrow \quad x^2 + 8x - 2x - 16 = 0$

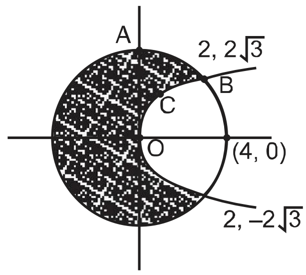
$\Rightarrow \quad x(x+8) - 2(x+8) = 0$

$\Rightarrow \qquad (x+8)(x-2) = 0$

$\qquad x = -8 \qquad$ (not possible $\because y^2$ can not be – ve)

or $\qquad x = 2 \qquad$ (only allowed value)

$\therefore \qquad y = \pm 2\sqrt{3}$



Area of $OABCO = \displaystyle\int_0^{2\sqrt{3}} \left(\sqrt{16 - y^2} - \frac{y^2}{6}\right) dy$

$$= \left[\frac{y}{2}\sqrt{16 - y^2} + \frac{16}{2}\sin^{-1}\frac{y}{4} - \frac{y^3}{18}\right]_0^{2\sqrt{3}}$$

$$\left[\int \sqrt{a^2 - x^2} = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \left[\sqrt{3}\,.\sqrt{16 - 12} + 8\sin^{-1}\frac{\sqrt{3}}{2} - \frac{24\sqrt{3}}{18}\right]$$

$$= \left[\sqrt{3}\,.\,2 + 8\frac{\pi}{3} - \frac{4}{\sqrt{3}}\right] = 2\sqrt{3} - \frac{4}{\sqrt{3}} + \frac{8}{3}\pi = \frac{2}{3}\sqrt{3} + \frac{8}{3}\pi$$

$\therefore \quad$ Required are $= 2\left(\dfrac{2\sqrt{3}}{3} + \dfrac{8}{3}\pi\right) + \dfrac{1}{2}(\pi 4^2)$

$$= \frac{4\sqrt{3}}{3} + \frac{16}{3}\pi + 8\pi = \frac{4\sqrt{3}}{3} + \frac{40}{3}\pi$$

$$= \frac{4}{3}(\sqrt{3} + 10\pi) \text{ sq. units}$$

26. $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$ ...(*i*)

Using property $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$, we have

$$\therefore \quad I = \int_0^{\pi} \frac{(\pi - x) \tan (\pi - x)}{\sec (\pi - x) + \tan (\pi - x)}\, dx$$

$$I = \int_0^{\pi} \frac{(\pi - x)(-\tan x)}{-\sec x - \tan x}\, dx$$

$$I = \int_0^{\pi} \frac{\pi \,.\tan x}{\sec x + \tan x}\, dx - \int_0^{\pi} \frac{x \,.\tan x}{\sec x + \tan x}\, dx \quad ...(ii)$$

Adding (*i*) and (*ii*) we have

$$2I = \pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x}\, dx$$

$$[f(x) = f(2a - x)] \text{ then } \int_0^{2a} f(x)\, dx = 2 \,. \int_0^a f(x)\, dx$$

$$\Rightarrow \quad 2I = \pi \times 2 \times \int_0^{\pi/2} \frac{\sin x}{1 + \sin x}\, dx$$

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} \frac{\sin x + 1 - 1}{1 + \sin x}\, dx$$

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} dx - \pi \int_0^{\pi/2} \frac{1}{1 + \sin x}\, dx$$

$$\Rightarrow \quad I = \pi \frac{\pi}{2} - \pi \int_0^{\pi/2} \frac{1}{1 + \cos x}\, dx \qquad \left[\text{Using} \int_0^a f(x) dx = \int_0^a f(a - x) dx\right]$$

$$\Rightarrow \quad I = \frac{\pi^2}{2} - \pi \int_0^{\pi/2} \frac{1}{2\cos^2 \frac{x}{2}}\, dx$$

$$\Rightarrow \quad I = \frac{\pi^2}{2} - \frac{\pi}{2} . \int_0^{\pi/2} \sec^2 \frac{x}{2} . dx$$

$$\Rightarrow \quad I = \frac{\pi^2}{2} - \frac{\pi}{2} . \left[\frac{\tan \frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi/2}$$

$$I = \frac{\pi^2}{2} - \frac{\pi}{2} \times 2 \times \left[\tan \frac{\pi}{4} - \tan 0\right]$$

$$I = \frac{\pi^2}{2} - \pi$$

27. Let $\dfrac{x+2}{3} = \dfrac{2y+3}{4} = \dfrac{3z+4}{5} = \lambda$

Any general point on the line is

$$3\lambda - 2, \quad \frac{4\lambda - 3}{2}, \quad \frac{5\lambda - 4}{3}$$

Now, direction ratio if a point on the line is joined to (– 2, 3, – 4) are

$$\Rightarrow \quad 3\lambda, \frac{4\lambda - 9}{2}, \frac{5\lambda + 8}{3}$$

Now the distance is measured parallel to the plane

$$4x + 12y - 3z + 1 = 0$$

$$\therefore \quad 4 \times 3\lambda + 12 \times \left(\frac{4\lambda - 9}{2}\right) - 3 \times \left(\frac{5\lambda + 8}{3}\right) = 0$$

$$\Rightarrow \quad 12\lambda + 24\lambda - 54 - 5\lambda - 8 = 0$$

$$31\lambda - 62 = 0$$

$$\Rightarrow \quad \lambda = 2$$

$\therefore$ The point required is $\left(4, \dfrac{5}{2}, 2\right)$.

$$\therefore \quad \text{Distance} = \sqrt{(4+2)^2 + \left(\frac{5}{2} - 3\right)^2 + (2+4)^2}$$

$$= \sqrt{36 + 36 + \frac{1}{4}} = \sqrt{\frac{289}{4}} = \frac{17}{2} \text{ units}$$

28. Let there be x tickets of first class and y tickets of second class. Then the problem is to

$$\max \; z = 400x + 300y$$

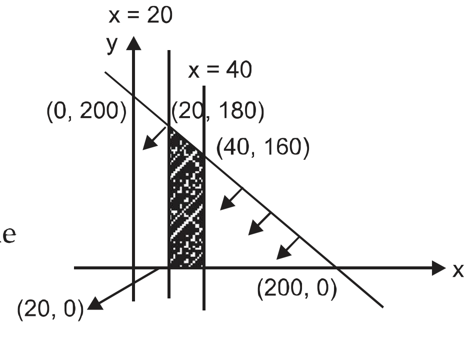
Subject to $x + y \leq 200$

$$x \geq 20$$

$$x + 4x \leq 200$$

$$5x \leq 200$$

$$x \leq 40$$



The shaded region in the graph represents the feasible region which is proved.

Le us evaluate the value of z at each corner point

z at $(20, 0), z = 400 \times 20 + 300 \times 0 = 8000$

z at $(40, 0) = 400 \times 40 + 300 \times 0 = 16000$

z at $(40, 160) = 400 \times 40 + 300 \times 160 = 16000 + 48000 = 64000$

z at $(20, 180) = 400 \times 20 + 300 \times 180 = 8000 + 54000 = 62000$

$$\max \; z = 64000 \text{ for } x = 40, y = 160$$

$\therefore$ 40 tickets of first class and 160 tickets of second class should be sold to earn maximum profit of Rs. 64,000.

29. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six. [CBSE 2005]

Sol. Let E be the event that the man reports that six occurs in the throwing of the die and let S_1 be the event that six occurs and S_2 be the event that six does not occur.

Then $P(S_1)$ = Probability that *six* occurs $= \frac{1}{6}$

$P(S_2)$ = Probability that *six* does not occur $= \frac{5}{6}$

$P(E/S_1)$ = Probability that the man reports that *six* occurs when *six* has actually occurred on the die

= Probability that the man speaks the truth $= \frac{3}{4}$

$P(E/S_2)$ = Probability that the man reports that *six* occurs when *six* has not actually occurred on the die

= Probability that the man does not speak the truth $= 1 - \frac{3}{4} = \frac{1}{4}$.

Thus, by Bayes' theorem, we get

$P(S_1/E)$ = Probability that the report of the man that *six* has occurred is actually a *six*

$$= \frac{P(S_1)\,P(E/S_1)}{P(S_1)\,P(E/S_1) + P(S_2)\,P(E/S_2)} = \frac{\frac{1}{6} \times \frac{3}{4}}{\frac{1}{6} \times \frac{3}{4} + \frac{5}{6} \times \frac{1}{4}} = \frac{3}{8}.$$

Set–II

20. Let $\Delta = \begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$, we have

$$\Delta = \begin{vmatrix} 3(a+b) & 3(a+b) & 3(a+b) \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

Taking out $3(a+b)$ from 1st row, we have

$$\Delta = 3(a+b) \begin{vmatrix} 1 & 1 & 1 \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

$$\Delta = 3(a+b)\begin{vmatrix} 0 & 0 & 1 \\ 2b & -b & a+b \\ -b & 2b & a \end{vmatrix}$$

Expanding along first row, we have

$$\Delta = 3(a+b)\,[1.(4b^2 - b^2)]$$
$$= 3\,(a+b) \times 3b^2 = 9b^2\,(a+b)$$

21. Let $I = \int_0^{\pi/2} \log \sin x \, dx$ ...(*i*)

$\Rightarrow \quad I = \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$

$\Rightarrow \quad I = \int_0^{\pi/2} \log \cos x \, dx$ ...(*ii*)

Adding (*i*) and (*i*) we have,

$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx$

$\Rightarrow \quad 2I = \int_0^{\pi/2} \log \sin x \cos x \, dx$

$\Rightarrow \quad 2I = \int_0^{\pi/2} \log \frac{2 \sin x \cos x}{2} \, dx$

$\Rightarrow \quad 2I = \int_0^{\pi/2} (\log \sin 2\,x - \log 2)\, dx$

$\Rightarrow \quad 2I = \int_0^{\pi/2} \log \sin 2x \, dx - \int_0^{\pi/2} \log 2 dx$

Let $2x = t \quad \Rightarrow \quad dx = \frac{dt}{2}$

When $\quad x = 0, \frac{\pi}{2}, t = 0, \pi$

$\therefore \quad 2I = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt - \log 2.\left(\frac{\pi}{2} - 0\right)$

$\Rightarrow \quad 2I = I - \frac{\pi}{2} \log 2$ $\quad \left[\because \int_0^{a} f(x)\, dx = \int_0^{a} f(t)\, dtx\right]$

$\Rightarrow \quad 2I - I = -\frac{\pi}{2} \log 2$

$\Rightarrow \quad I = -\frac{\pi}{2} \log 2$

22. We have

$(1 + x^2)\frac{dy}{dx} + y = \tan^{-1} x$

Dividing each term by $(1 + x^2)$

$$\frac{dy}{dx}+\frac{1}{1+x^2}.y=\frac{\tan^{-1}x}{1+x^2}$$

Clearly, it is linear differential equation of the form $\frac{dy}{dx}+P.y=Q$

So, $P=\frac{1}{1+x^2}$ and $Q=\frac{\tan^{-1}x}{1+x^2}$

$\therefore$ Integrating factor, I. F. $=e^{\int P\,dx}=e^{\int\frac{1}{1+x^2}dx}=e^{\tan^{-1}x}$

Therefore, solution of given differential equation is

$$y\times I.F.=\int Q\times I.F.\,dx$$

$$\Rightarrow\quad y.e^{\tan^{-1}x}=\int\frac{\tan^{-1}x}{1+x^2}.e^{\tan^{-1}x}dx$$

Let $\quad I=\int\frac{\tan^{-1}x\,e^{\tan^{-1}x}}{1+x^2}dx$

Let $\quad e^{\tan^{-1}x}=t \quad\Rightarrow\quad \frac{e^{\tan^{-1}x}}{1+x^2}dx=dt$

Also $\quad \tan^{-1}x=\log t$

$\Rightarrow\quad I=\int\log t\,dt$

$\Rightarrow\quad I=t\log t-t+C$ [Integrating by parts]

$\Rightarrow\quad I=e^{\tan^{-1}x}.\tan^{-1}x-e^{\tan^{-1}x}+C$

Hence required solution is

$$y.e^{\tan^{-1}x}=e^{\tan^{-1}x}(\tan^{-1}x-1)+C$$

$$\Rightarrow\quad y=(\tan^{-1}x-1)+C\,e^{-\tan^{-1}x}$$

27. The given system of linear equations.

$$3x-2y+3z=8$$
$$2x+y-z=1$$
$$4x-3y+2z=4$$

We write the system of linear equation in matrix form

$$\begin{bmatrix}3&-2&3\\2&1&-1\\4&-3&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}8\\1\\4\end{bmatrix}$$

$$\Rightarrow \quad A.X = B, \text{ where } A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1} B$$

Now, co-factors of matrix A are

$$C_{11} = (-1)^{1+1}.(2-3) = (-1)^2 .(-1) = -1$$

$$C_{12} = (-1)^{1+2}.(4+4) = (-1)^3 .8 = -8$$

$$C_{13} = (-1)^{1+3}.(-6-4) = (-1)^4 .(-10) = -10$$

$$C_{21} = (-1)^{2+1}(-4+9) = (-1)^3 (5) = -5$$

$$C_{22} = (-1)^{2+2}.(6-12) = (-1)^4 (-6) = -6$$

$$C_{23} = (-1)^{2+3}(-9+8) = (-1)^5 (-1) = 1$$

$$C_{31} = (-1)^{3+1}(2-3) = (-1)^4 (-1) = -1$$

$$C_{32} = (-1)^{3+2}(-3-6) = (-1)^5 .(-9) = 9$$

$$C_{33} = (-1)^{3+3} (3+4) = (-1)^6 7 = 7$$

$$\therefore \quad \text{adj } A = c^T = \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$ Where c = matric of co-factors of elements.

and $$|A| = \begin{vmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{vmatrix} = 3(2-3) + 2(4+4) + 3(-6-4)$$

$$= 3 \times -1 + 2 \times 8 + 3 \times -10 = -3 + 16 - 30 = -17$$

$$\therefore A^{-1} = \frac{adj\, A}{|A|} = -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

Now, $X = A^{-1} B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -8 & -5 & -4 \\ -64 & -6 & +36 \\ -80 & +1 & +28 \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore \quad x = 1, y = 2, z = 3$$

OR

For elementary transformation we have, $A = IA$

$$\Rightarrow \quad \begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 1 \\ 1 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_3$

$$\Rightarrow \quad \begin{bmatrix} 1 & -1 & 1 \\ 3 & 4 & 1 \\ 1 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - R_1$

$$\Rightarrow \quad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 7 & -2 \\ 0 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix} A$$

Applying $R_2 \to \frac{4}{7} R_2$

$$\Rightarrow \quad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & \frac{-2}{7} \\ 0 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ \frac{-3}{7} & \frac{1}{7} & \frac{3}{7} \\ -1 & 0 & 2 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & \frac{5}{7} \\ 0 & 1 & \frac{-2}{7} \\ 0 & 7 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} & \frac{-4}{7} \\ \frac{-3}{7} & \frac{1}{7} & \frac{3}{7} \\ -1 & 0 & 2 \end{bmatrix} A$$

Applying $R_3 \to R_3 + 7R_2$

$$\begin{bmatrix} 1 & 0 & \frac{5}{7} \\ 0 & 1 & \frac{-2}{7} \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} & \frac{-4}{7} \\ \frac{-3}{7} & \frac{1}{7} & \frac{3}{7} \\ 2 & -1 & -1 \end{bmatrix} A$$

Applying $R_3 \to \frac{R_3}{3}$

$$\begin{bmatrix} 1 & 0 & \frac{5}{7} \\ 0 & 1 & \frac{-2}{7} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} & \frac{-4}{7} \\ \frac{-3}{7} & \frac{1}{7} & \frac{3}{7} \\ \frac{-2}{3} & \frac{-1}{3} & \frac{-1}{3} \end{bmatrix} A$$

$R_1 \to R_1 \frac{-5}{7} R_3, \quad R_2 \to R_2 + \frac{2}{7} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{21} & \frac{8}{21} & \frac{-1}{3} \\ \frac{-5}{21} & \frac{1}{21} & \frac{1}{3} \\ \frac{-2}{3} & \frac{-1}{3} & \frac{-1}{3} \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{2}{21} & \frac{8}{21} & \frac{-1}{3} \\ \frac{-5}{21} & \frac{1}{21} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} & \frac{-1}{3} \end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} 2 & 8 & -7 \\ -5 & 1 & 7 \\ +14 & -7 & -7 \end{bmatrix}$$

28. Let

S = Event of insurance of scooter driver

C = Event of insurance of Car driver

T = Event of insurance of Truck driver

and A = Event of meeting with an accident

Now, we have, $P(S) =$ Probability of insurance of scooter driver

$\Rightarrow P(S) = \frac{2000}{9000} = \frac{2}{9}$

$P(C)$ = Probability of insurance of car driver

$\Rightarrow P(C) = \frac{3000}{9000} = \frac{3}{9}$

$P(T)$ = Probability of insurance of Truck driver

$\Rightarrow P(T) = \frac{4000}{9000} = \frac{4}{9}$

and, $P(A / S)$ = Probability that scooter driver meet. with an accident

$\Rightarrow P(A / S) = 0.04$

$P(A / C)$ = Probability that car driver meet with an accident

$\Rightarrow P(A / C) = 0.06$

$P(A / T)$ = Probability that Truck driver meet with an accident

$\Rightarrow P(A / T) = 0.15$

By Baye's theorem, we have the required probability

$$P(C/A) = \frac{P(C).P(A/C)}{P(S).P(A/S) + P(C)\,P(A/C) + P(T).P(A/T)}$$

$$= \frac{\frac{3}{9} \times 0.06}{\frac{2}{9} \times 0.04 + \frac{3}{9} \times 0.06 + \frac{4}{9} \times 0.15}$$

$$= \frac{3 \times 0.06}{2 \times 0.04 + 3 \times 0.06 + 4 \times 0.15} = \frac{0.18}{0.08 + 0.18 + 0.60}$$

$$= \frac{0.18}{0.86} = \frac{18}{86} = \frac{9}{43}$$

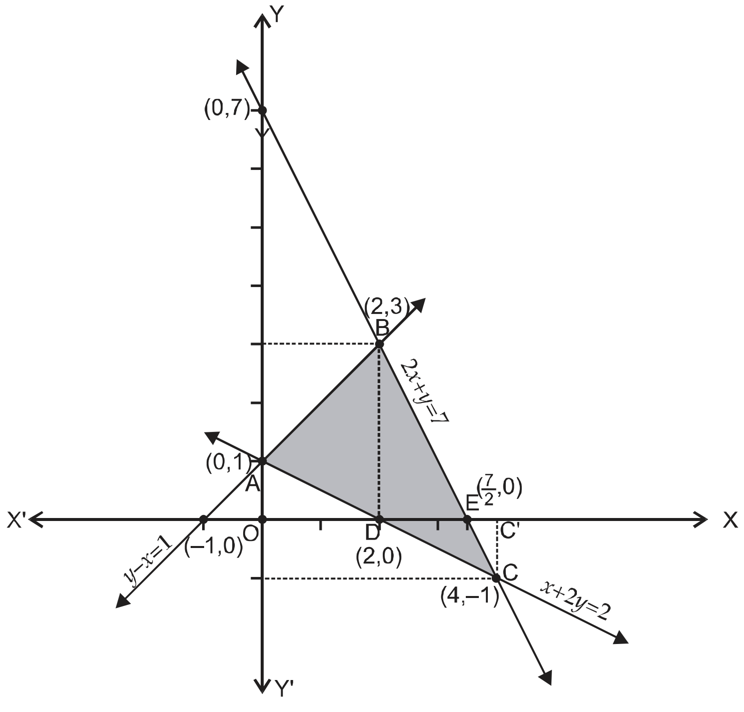
29. Given,

$$x + 2y = 2 \qquad ...(i)$$

$$y - x = 1 \qquad ...(ii)$$

$$2x + y = 7 \qquad ...(iii)$$

On plotting these lines, we have



Area of required region

$$= \int_{-1}^{3} \frac{7-y}{2}\,dy - \int_{-1}^{1} (2-2y)\,dy - \int_{1}^{3} (y-1)\,dy$$

$$= \frac{1}{2}\left[7y - \frac{y^2}{2}\right]_{-1}^{3} - [2y - y^2]_{-1}^{1} - \left[\frac{y^2}{2} - y\right]_{1}^{3}$$

$$= \frac{1}{2}\left(21 - \frac{9}{2} + 7 + \frac{1}{2}\right) - (2 - 1 + 2 + 1) - \left(\frac{9}{2} - 3 - \frac{1}{2} + 1\right)$$

$= 12 - 4 - 2 = 6$ sq. units

Set–III

20. We have

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we have

$$\Delta = \begin{vmatrix} (a+b+c) & b & c \\ (a+b+c) & c & a \\ (a+b+c) & a & b \end{vmatrix}$$

taking out $(a+b+c)$ from Ist column, we have

$$\Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Interchanging column into row, we have

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we have

$$\Delta = (a+b+c)\begin{vmatrix} 0 & 0 & 1 \\ b-c & c-a & a \\ c-a & a-b & b \end{vmatrix}$$

Expanding along Ist row, we have

$$\Delta = (a+b+c)\,[1\,(b-c)(a-b) - (c-a)^2]$$

$$= (a+b+c)(ba - b^2 - ca + bc - c^2 - a^2 + 2ac)$$

$\Rightarrow \quad \Delta = (a+b+c)(ab + bc + ca - a^2 - b^2 - c^2)$

$\Rightarrow \quad \Delta = -(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$\Rightarrow \quad \Delta = -\frac{1}{2}(a+b+c)\{(a-b)^2 + (b-c)^2 + (c-a)^2\}$

Here, $(a+b+c)$ is positive as a, b, c are all positive
and it is clear that $(a-b)^2 + (b-c)^2 + (c-a)^2$ is also positive

Hence $\quad \Delta = -\frac{1}{2}(a+b+c)\,[(a-b)^2 + (b-c)^2 + (c-a)^2]$ has negative value.

21. Let $I = \int_0^1 \cot^{-1}(1 - x + x^2)\, dx$

$= \int_0^1 \tan^{-1} \dfrac{1}{1 - x + x^2}\, dx$ $\left[\because \cot^{-1} x = \tan^{-1} \dfrac{1}{x}\right]$

$= \int_0^1 \tan^{-1} \dfrac{x + (1-x)}{1 - x(1-x)}\, dx$ [$\because$ 1 can be written as $x + 1 - x$]

$= \int_0^1 [\tan^{-1} x + \tan^{-1}(1-x)]\, dx$ $\left[\because \tan^{-1}\left\{\dfrac{a+b}{1-ab}\right\} = \tan^{-1} a + \tan^{-1} b\right]$

$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1-x)\, dx$

$= \int_0^1 \tan^{-1} x dx + \int_0^1 \tan^{-1}[1 - (1-x)]\, dx$ $\left[\because \int_0^a f(x) = \int_0^a f(a-x)\, dx\right]$

$= 2\int_0^1 \tan^{-1} x dx = 2\int_0^1 \tan^{-1} x . 1\, dx,$ integrating by parts, we get

$= 2\left[\{\tan^{-1} x . x\}_0^1 - \int_0^1 \dfrac{1}{1+x^2} \cdot x dx\right]$

$= 2[\tan^{-1} 1 - 0] - \int_0^1 \dfrac{2x}{1+x^2}\, dx = 2 \cdot \dfrac{\pi}{4} - [\log(1+x^2)]_0^1$

$= \dfrac{\pi}{2} - (\log 2 - \log 1) = \dfrac{\pi}{2} - \log 2$ [$\because \log 1 = 0$]

22. We have the differential equation

$$x \log x \frac{dy}{dx} + y = 2 \log x$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{x \log x} . y = \frac{2}{x}$$

It is linear differential equation of the from $\dfrac{dy}{dx} + Py = Q$

So, Here $P = \dfrac{1}{x \log x}$ and $Q = \dfrac{2}{x}$

Now, I.F. $= e^{\int p dx} = e^{\int \frac{1}{x \log x} dx} = e^{\log |\log x|}$

$= \log x$

Hence, solution of given differential equation is $y \times I.F. = \int Q \times I.F\, dx$

$\Rightarrow \quad y \log x = \int \frac{2}{x} . \log x \, dx$

$\Rightarrow \quad y \log x = 2 \int \frac{1}{x} . \log x \, dx = 2. \frac{(\log x)^2}{2} + C$

$\Rightarrow \quad y \log x = (\log x)^2 + C$

27. The given system of linear equations is

$$x + y + z = 4$$
$$2x + y - 3z = -9$$
$$2x - y + z = -1$$

We write the system of equation in Matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \\ -1 \end{bmatrix}$$

$\Rightarrow \quad AX = B$, we have

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ 2 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ -9 \\ -1 \end{bmatrix}$$

$\therefore \quad X = A^{-1} B$

Now, co-factors of A

$C_{11} = (-1)^{1+1} (1 - 3) = -2;$ $\quad C_{12} = (-1)^{1+2} (2 + 6) = -8$

$C_{13} = (-1)^{1+3} (-2 - 2) = -4;$ $\quad C_{21} = (-1)^{2+1} (1 + 1) = -2$

$C_{22} = (-1)^{2+2} (1 - 2) = -1;$ $\quad C_{23} = (-1)^{2+3} (-1 - 2) = 3$

$C_{31} = (-1)^{3+1} (-3 - 1) = -4;$ $\quad C_{32} = (-1)^{3+2} (-3 - 2) = 5$

$C_{33} = (-1)^{3+3} = (1 - 2) = -1$

$$\therefore \quad adj\, A = (C)^T = \begin{bmatrix} -2 & -2 & -4 \\ -8 & -1 & 5 \\ -4 & 3 & -1 \end{bmatrix}$$

Now, $|A| = 1(-2) - 1(8) + 1(-4)$

$= -2 - 8 - 4 = -14$

$$\therefore \quad A^{-1} = \frac{adj.\, A}{|A|}$$

$$= \frac{\begin{bmatrix} -2 & -2 & -4 \\ -8 & -1 & 5 \\ -4 & 3 & -1 \end{bmatrix}}{-14} = \frac{1}{14} \begin{bmatrix} 2 & 2 & 4 \\ 8 & 1 & -5 \\ 4 & -3 & 1 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = .\frac{1}{14}\begin{bmatrix} 2 & 2 & 4 \\ 8 & 1 & -5 \\ 4 & -3 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -9 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 8+(-18) & +(-4) \\ 32+(-9) & +5 \\ 16+\ \ 27 & +(-1) \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{14}\begin{bmatrix} -14 \\ 28 \\ 42 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore \quad x = -1, y = 2$ and $z = 3$ is the required solution.

OR

Let $\quad A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 1 \\ 1 & 6 & 3 \end{bmatrix}$

Therefore, for elementary row transformation, we have

$A = I\,A$

$$\Rightarrow \quad \begin{bmatrix} 2 & 5 & 3 \\ 3 & 4 & 1 \\ 1 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & -1 & 0 \\ 3 & 4 & 1 \\ 1 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 7 & 1 \\ 1 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 7 & 1 \\ 0 & 7 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix} A$$

Applying $R_1 \to R_1 + \frac{1}{7}R_2$

$$\begin{bmatrix} 1 & 0 & \frac{1}{7} \\ 0 & 7 & 1 \\ 0 & 7 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} & \frac{-4}{7} \\ -3 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix} A$$

Applying $R_2 \to \frac{R_2}{7}$

$$\begin{bmatrix} 1 & 0 & \frac{1}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 7 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} & \frac{-4}{7} \\ \frac{-3}{7} & \frac{1}{7} & \frac{3}{7} \\ -1 & 0 & 2 \end{bmatrix} A$$

Applying $R_3 \to R_3 - 7R_2$

$$\begin{bmatrix} 1 & 0 & \frac{1}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} & \frac{-4}{7} \\ \frac{-3}{7} & \frac{1}{7} & \frac{3}{7} \\ 2 & -1 & -1 \end{bmatrix} A$$

Applying $R_3 \to \frac{R_3}{2}$

$$\begin{bmatrix} 1 & 0 & \frac{1}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} & \frac{-4}{7} \\ \frac{-3}{7} & \frac{1}{7} & \frac{3}{7} \\ 1 & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix} A$$

Applying $R_1 \to R_1 - \frac{1}{7} R_3, R_2 \to R_2 - \frac{1}{7} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{3}{14} & \frac{-1}{2} \\ \frac{-4}{7} & \frac{3}{14} & \frac{1}{2} \\ 1 & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix} A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{3}{7} & \frac{3}{14} & \frac{-1}{2} \\ \frac{-4}{7} & \frac{3}{14} & \frac{1}{2} \\ 1 & \frac{-1}{2} & \frac{-1}{2} \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 6 & 3 & -7 \\ -8 & 3 & 7 \\ 14 & -7 & -7 \end{bmatrix}$$

28. The equations of the given curves are

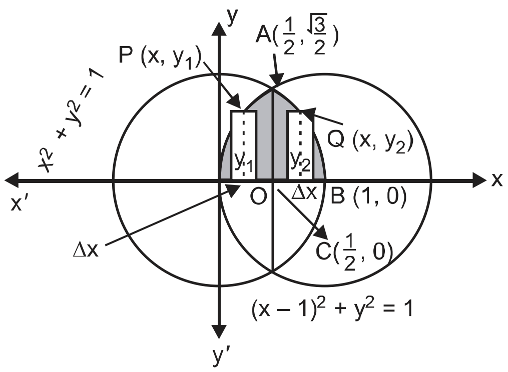
$$x^2 + y^2 = 1 \qquad \ldots(i)$$

and, $(x-1)^2 + (y-0)^2 = 1$ ...(*ii*)

Clearly, $x^2 + y^2 = 1$ represents a circle with centre at (0, 0) and radius unity. Also, $(x-1)^2 + y^2 = 1$ represents a circle with centre at (1, 0) and radius unity. To find the points of intersection of the given curves, we solve (1) and (2) simultaneously.

Thus, $1-(x-1)^2 = 1-x^2$

$\Rightarrow \quad 2x = 1 \quad \Rightarrow \quad x = \dfrac{1}{2}$



We find that the two curves intersect at $A(1/2, \sqrt{3}/2)$ and $D(1/2, -\sqrt{3}/2)$.

Since both the curves are symmetrical about x-axis.

So, Required area = 2 (Area *OABCO*)

Now, we slice the area *OABCO* into vertical strips. We observe that the vertical strips change their character at $A(1/2, \sqrt{3}/2)$. So.

Area *OABCO* = Area *OACO* + Area *CABC*.

When area *OACO* is sliced into vertical strips, we find that each strip has its upper end on the circle $(x-1)^2 + (y-0)^2 = 1$ and the lower end on x-axis. So, the approximating rectangle shown in Fig. has, Length = y_1, Width = Δx and Area = $y_1\,\Delta x$. As it can move from $x = 0$ to $x = 1/2$.

$\therefore$ Area $OACO = \int_0^{1/2} y_1 dx$

$\Rightarrow$ Area $OACO = \int_0^{1/2} \sqrt{1-(x-1)^2}\, dx \quad \left[\begin{array}{l}\because P(x, y_1) \text{ lies on } (x-1)^2 + y^2 = 1 \\ \therefore (x-1)^2 + y_1^2 = 1 \Rightarrow y_1 = \sqrt{1-(x-1)^2}\end{array}\right]$

Similarly, approximating rectangle in the region *CABC* has, Length, = y_2, Width Δx and Area = $y_2 \Delta x$. As it can move form $x = \dfrac{1}{2}$ to $x = 1$.

$\therefore$ Area $CABC = \int_{1/2}^{1} y_2\, dx$

$\Rightarrow$ Area $CABC = \int_{1/2}^{1} \sqrt{1-x^2}\, dx \quad \left[\begin{array}{l}\because Q(x, y_2) \text{ lies on } x^2 + y^2 = 1 \\ \therefore x^2 + y_2^2 = 1 \Rightarrow y_2 = \sqrt{1-x^2}\end{array}\right]$

Hence, required area A is given by

$$A = 2\left[\int_0^{1/2} \sqrt{1-(x-1)^2}\, dx + \int_{1/2}^{1} \sqrt{1-x^2}\, dx\right]$$

$$\Rightarrow \quad A = 2\left[\left[\frac{1}{2}.(x-1)\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-1}{1}\right)\right]_0^{1/2} + \left[\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_{1/2}^{1}\right]$$

$$\Rightarrow \quad A = \left[\left\{-\frac{\sqrt{3}}{4} + \sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(-1)\right\} + \left\{\sin^{-1}(1) - \frac{\sqrt{3}}{4} - \sin^{-1}\left(\frac{1}{2}\right)\right\}\right]$$

$$\Rightarrow \quad A = -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{sq. units}$$

29. Let

S = Event of insuring scooter driver

C = Event of insuring Car driver

T = Event of insuring Truck driver

and A = Event of meeting with an accident.

Now, we have

$P(S)$ = Probability of insuring scooter driver $= \frac{3000}{15000} = \frac{3}{15}$

$P(C)$ = Probability of insuring car driver $= \frac{5000}{15000} = \frac{5}{15}$

$P(T)$ = Probability of insuring Truck driver $= \frac{7000}{15000} = \frac{7}{15}$

and, $P(A / S)$ = Probability that scooter driver meet with an accident = 0.04

$P(A / C)$ = Probability that car driver meet with an accident = 0.05

$P(A / T)$ = Probability that Truck driver meet with an accident = 0.15

By Baye's theorem, we have

Required probability $= P(C / A) = \dfrac{P(C).P(A / C)}{P(S).P(A / S) + P(C).P(A / C) + P(T).P(A / T)}$

$$= \frac{\frac{5}{15} \times 0.05}{\frac{3}{15} \times 0.04 + \frac{5}{15} \times 0.05 + \frac{7}{15} \times 0.15}$$

$$= \frac{5 \times 0.05}{3 \times 0.04 + 5 \times 0.05 + 7 \times 0.15}$$

$$= \frac{0.25}{0.12 + 0.25 + 1.05}$$

$$= \frac{0.25}{1.42} = \frac{25}{142}$$